INVARIANT MEASURES FOR TRAIN TRACK TOWERS

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ABSTRACT. In this paper we present a combinatorial machinery, consisting of a *graph tower* $\overleftarrow{\Gamma}$ and a *weight towers* $\overleftarrow{\varpi}$ on $\overleftarrow{\Gamma}$, which allow us to efficiently describe invariant measures $\mu = \mu^{\overleftarrow{\varpi}}$ on rather general discrete dynamicals system over a finite alphabet.

A train track map $f : \Gamma \to \Gamma$ defines canonically a *stationary* such graph tower $\overleftarrow{\Gamma}_f$. In the most important two special cases the measure μ specializes to a (typically ergodic) invariant measure on a substitution subshift, or to a projectively f_*-invariant current on the free group $\pi_1\Gamma$. Our main result establishes a 1-1 correspondence between such measures μ and the non-negative eigenvectors of the incidence ("transition") matrix of f.

1. Introduction

The goal of this paper is to present a rather general graph theoretic method to describe invariant measures on discrete dynamical systems over a finite alphabet. The novelty of this method is underlined by the fact that it doesn't use Bratteli diagrams, Rokhlin towers or any other of the established methods to describe such measures. We work with so called *train track maps* $f : \Gamma \to \Gamma$, i.e. f is a self-map of a connected graph Γ that maps vertices to vertices and edges e to reduced edge paths $f(e)$, where f has in addition the crucial property that for any exponent $t \geqslant 1$ the t-th iterate image path $f^t(e)$ is still reduced. To any such train track map there is canonically associated an *infinitely legal lamination* $L_\infty(f)$ which is a set of biinfinite reduced edge paths in Γ on which (under a mild technical *non-repeating* assumption) f acts bijectively.

Our main result, stated below in detail, can be paraphrased slightly by stating that we exhibit a natural bijection between the non-negative eigenvectors of the incidence matrix of f with eigenvalues $\lambda > 1$ on one hand, and finitary invariant measures μ on $L_\infty(f)$ which satisfy $f_*\mu = \lambda\mu$ on the other.

In the special case where Γ is a connected 1-vertex graph (a "rose"), and where for a suitable orientation on the edges e_i the image paths $f(e_i)$ cross only over positively oriented edges, our setting amounts to what is known in symbolic dynamics under the name of *substitutions*: in this case the *subshift* defined by the substitution essentially coincides with the infinitely legal lamination $L_\infty(f)$.

The other important special case, where we assume that f is a homotopy equivalence, brings us into the world of graphs Γ provided with a marking $\pi_1\Gamma \overset{\cong}{\to} F_N$, and of outer automorphisms φ of the non-abelian free group F_N of finite rank $N \geqslant 2$ which are represented (via the marking isomorphism) by such a train track map f. In this case, finitary invariant measures on $L_\infty(f)$ are known as *currents* on F_N: The projectivized space $\mathbb{P}\mathrm{Curr}(F_N)$ of such currents is known to be compact, and although it is infinite dimensional, the natural action of $\mathrm{Out}(F_N)$ on $\mathbb{P}\mathrm{Curr}(F_N)$ has remarkably strong similarities with the action of the mapping class group on Teichmüller space (see [14] and the references given there).

The main tool introduced and studied in this paper (see §4) are *graph towers*

$$\ldots \xrightarrow{f_{n+1,n+2}} \Gamma_{n+1} \xrightarrow{f_{n,n+1}} \Gamma_n \xrightarrow{f_{n-1,n}} \Gamma_{n-1} \xrightarrow{f_{n-2,n-1}} \ldots \xrightarrow{f_{0,1}} \Gamma_0 = \Gamma \tag{1.1}$$

given by infinitely many finite *level graphs* Γ_n, connected by graph maps $f_{n-1,n}$ which map edges to non-trivial reduced edge paths. Such a graph tower $\overleftarrow{\Gamma}$ is *expanding* if the length of the paths $f_{0,1}f_{1,2}\ldots f_{n-1,n}(\gamma_n)$ tends with n to ∞, for any choice of non-trivial edge paths γ_n in Γ_n which connects two vertices that both have $\geqslant 3$ adjacent edges.

The graph tower $\overleftarrow{\Gamma}$ determines a language $\mathcal{P}_{legal}(\overleftarrow{\Gamma})$ which consists of all finite reduced paths γ in Γ that are *infinitely legal*, i.e. they are images of reduced paths from an arbitrary high level graph. In the usual fashion the language $\mathcal{P}_{legal}(\overleftarrow{\Gamma})$ generates a Cantor set $L_{legal}(\overleftarrow{\Gamma})$, called the *infinitely legal lamination* of $\overleftarrow{\Gamma}$, which consists of biinfinite paths and is naturally equipped with a shift map.

By putting (see §5) a non-negative weight function ω_n on the edges of each level graph Γ_n, such that the resulting *weight tower* $\overleftarrow{\omega} = (\omega_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfies certain natural compatibility conditions, one obtains a Kolmogorov function $\mu_\Gamma^{\overleftarrow{\omega}}$ on the language $\mathcal{P}_{legal}(\overleftarrow{\Gamma})$ and hence a finite measure on $L_{legal}(\overleftarrow{\Gamma})$ that is invariant under the shift map. Conversely (see §7), every invariant measure on $L_{legal}(\overleftarrow{\Gamma})$ comes from such a weight tower, and under a natural combinatorial *non-repeating* hypothesis (which is equivalent to stating that every biinfinite path from $L_{legal}(\overleftarrow{\Gamma})$ has precisely on reduced preimage path in each level graph), we obtain the following bijective relationship (see Propositions 5.4 and 7.4):

Proposition 1.1. *For every non-repeating expanding graph tower $\overleftarrow{\Gamma}$ as in (1.1) the relation*

$$\overleftarrow{\omega} \mapsto \mu_\Gamma^{\overleftarrow{\omega}}$$

defines a natural bijection between weight towers and invariant measures on $L_{legal}(\overleftarrow{\Gamma})$.

After having put in place this general machinery, we turn (see §9) to train track maps $f : \Gamma \to \Gamma$ and associate to such f a "stationary" graph tower $\overleftarrow{\Gamma}_f$, which satisfies $L_{legal}(\overleftarrow{\Gamma}_f) = L_\infty(f)$. If f is *expanding* (= no contracted or periodic edges), then $\overleftarrow{\Gamma}_f$ is expanding, and if f is *non-repeating* (see §10), then $\overleftarrow{\Gamma}_f$ is non-repeating. The latter is automatically true if f is a homotopy equivalence.

We show that, for such a stationary graph tower $\overleftarrow{\Gamma}_f$, every non-negative eigenvector $\vec{v}$ with eigenvalue $\lambda > 1$ of the *transition matrix* $M(f)$ (i.e. the non-negative incidence matrix naturally associated to f) defines canonically a *vector tower* $\overleftarrow{v} = (\frac{1}{\lambda^n}\vec{v})_{n\in\mathbb{N}\cup\{0\}}$. Such $\overleftarrow{v}$ in turn defines a weight tower $\overleftarrow{\omega}(\overleftarrow{v}) = (\omega_n)_{n\in\mathbb{N}\cup\{0\}}$, where the weight $\omega_n(e)$ on any edge e of Γ_n is given by the e-coordinate of the vector $\frac{1}{\lambda^n}\vec{v}$. The invariant measure on $L_\infty(f) = L_{legal}(\overleftarrow{\Gamma}_f)$ defined via Proposition 1.1 by the weight tower $\overleftarrow{\omega}(\overleftarrow{v})$ is denoted by $\mu_\Gamma^{\vec{v}}$. If Γ is provided with a marking $\pi_1\Gamma \xrightarrow{\cong} F_N$, then the resulting current on F_N is denoted by $\mu^{\vec{v}}$.

For the special but important case that f is a homotopy equivalence, we obtain (see Propositions 9.4 and 9.5):

Theorem 1.2. *Let $f : \Gamma \to \Gamma$ be an expanding train track map that represents $\varphi \in Out(F_N)$, with transition matrix $M(f)$.*

For any non-negative eigenvector $\vec{v}$ of $M(f)$ with eigenvalue $\lambda > 1$ there is a current $\mu^{\vec{v}} \in Curr(F_N)$ which has support in $L_\infty(f)$ and satisfies:

$$\varphi(\mu^{\vec{v}}) = \lambda\mu^{\vec{v}}$$

Conversely, for any current $\mu \in Curr(F_N)$, which has support in $L_\infty(f)$ and satisfies $\varphi(\mu) = \lambda\mu$ for some scalar $\lambda > 1$, there exists a non-negative eigenvector $\vec{v}$ of $M(f)$ with eigenvalue λ that satisfies:

$$\mu = \mu^{\vec{v}}$$

The hypothesis in the above theorem that the automorphism φ of F_N can be represented by an expanding train track map $f : \Gamma \to \Gamma$ is less restrictive than what may seem at first sight: Bestvina-Handel [2] showed that the most important class of *fully irreducible* such φ always satisfies this hypothesis. Work in progress of the third author on a more general train track technology ("α-train-tracks" [17]) indicate that with very minor modifications our technique (and hence the above theorem) may indeed apply to a very wide class of automorphisms φ of F_N, including for example all hyperbolic such φ.

In case that the train track map $f : \Gamma \to \Gamma$ is not a homotopy equivalence (in particular if f is non-injective or non-injective on conjugacy classes), there is in general no well defined naturally induced map on the current space of $\pi_1\Gamma$. In this case, however, we can consider $L_\infty(f)$ as "subshift" space as is typically done in the symbolic dynamics. If f is non-repeating, it still induces a homeomorphism on the shift-orbit space, so that for any (shift) invariant measure μ on $L_\infty(f)$ there is a well defined image measure $f_*\mu$ on $L_\infty(f)$. We prove (see Theorem 10.3):

Theorem 1.3. *Let $f : \Gamma \to \Gamma$ be an expanding non-repeating train track map (not necessarily a homotopy equivalence), and let $\lambda > 1$ be an eigenvalue of $M(f)$.*

There is a canonical bijection between the set $\mathcal{M}_\lambda(f)$ of finite invariant measures μ on $L_\infty(f)$ which satisfy $f_*\mu = \lambda\mu$ and the set $V_\lambda(f)$ of non-negative $M(f)$-eigenvectors $\vec{v}$ with eigenvector λ, given by:

$$\vec{v} \mapsto \mu := \mu^{\vec{v}}$$

There is an interesting strong similarity between the last theorem and results of S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak [3, 4] (inspired by work of F. Durand, B. Host and C. Skau [11] on Bratteli diagrams, see also [10] for a survey): a similar bijective relationship as the above map $\vec{v} \mapsto \mu^{\vec{v}}$ has been exhibited for substitutions in [4], using very different methods (Bratteli diagrams and the Vershik map, to name some key ingredients). There are, however, a number of subtle and quite interesting differences between our vice versa results. We present the main theorem of S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak in section 11, and we explain there some of the technical and also substantial differences.

A bit of history: Graph towers (called there "combinatorial train tracks") can be traced back to [16], but without mentioning of weights or measures. In this context one should also mention Rauzy and De Bruijn graphs.

More recently, a version of graph towers appear in work of T. Coulbois and A. Hilion [5]: Given an $\mathbb{R}$-tree T with dense orbits in the boundary of Outer space, they use Rips induction to build a graph tower $\overleftarrow{\Gamma}$ such that $L_{legal}(\overleftarrow{\Gamma})$ is the dual lamination of T. In the case where Rips induction fails, they establish in joint work with P. Reynolds [9] a different kind of induction (in the spirit of Rauzy-Veech induction for IET), which also leads to a graph tower.

Ergodic properties of the lamination $L_{legal}(\overleftarrow{\Gamma})$ in this $\mathbb{R}$-tree context are studied in [6] via weight functions on $\overleftarrow{\Gamma}$; for generalisations see also the recent work of H. Namazi, A. Pettet and P. Reynolds [19].

Finally, as a "very recent" appearance we'd like to point to the joint work of M. Lustig and C. Uyanik [18], where the dynamics of hyperbolic automorphisms on current space are investigated, and where some of the results presented here are recovered by rather different methods.

2. Preliminaries

In this section we collect some basics and notation about graphs, graph maps, free groups, free group automorphisms, symbolic dynamics, etc. They will then be used freely in the subsequent sections.

2.1. **Graphs, edge paths, languages.**

In this paper a graph Γ is a topological (or combinatorial) space consisting of vertices v or v_i and non-oriented edges E or E_i. Since for practical purposes one almost always needs to work with oriented edges, we associate to every non-oriented edge E of Γ abstractly a pair of oppositely oriented edges, so that the set $\text{Edges}^{\pm}(\Gamma)$ of *oriented edges of* Γ contains twice as many elements than (non-oriented) edges present in the topological space Γ.

For every (oriented) edge $e \in \text{Edges}^{\pm}(\Gamma)$ we denote the edge in $\text{Edges}^{\pm}(\Gamma)$ with reversed orientation by $\overline{e}$, and of course one has $\overline{\overline{e}} = e$. The map $e \mapsto \overline{e}$ is hence a fixpoint-free involution on the set $\text{Edges}^{\pm}(\Gamma)$ of oriented edges of Γ. Whenever need be, we let

$$\text{Edges}^{+}(\Gamma) \subseteq \text{Edges}^{\pm}(\Gamma)$$

denote any section of the quotient map

$$\text{Edges}^{\pm}(\Gamma) \to \text{Edges}^{\pm}(\Gamma)/\langle e = \overline{e}\rangle\,.$$

We denote the *terminal endpoint* of an edge e by $\tau(e)$.

Unless otherwise stated, we always assume that a graph is finite (= finitely many edges and vertices), and that it is connected.

An edge path $\gamma = \ldots e_{i-1}e_ie_{i+1}\ldots$ is a finite, one-sided infinite or biinfinite sequence of edges $e_i \in \text{Edges}^{\pm}(\Gamma)$ such that $\tau(e_i) = \tau(\overline{e}_{i+1})$ for all indices i occurring in γ. Of course, the indexing is immaterial: for example, the paths $e_1e_2e_3$ and $e_4e_5e_6$ are equal if $e_1 = e_4, e_2 = e_5$ and $e_3 = e_6$.

For any edge path γ we denote by $\overline{\gamma}$ the inversely oriented path, i.e. for γ as above one has $\overline{\gamma} = \ldots\overline{e}_{i+1}\overline{e}_i\overline{e}_{i-1}\ldots$. The *combinatorial length* (or simply *length*) $|\gamma|$ of a finite path γ is equal to the number edges traversed by γ.

In general, an edge path $\gamma = \ldots e_{i-1}e_ie_{i+1}\ldots$ need not be *reduced*: it may well be that one has $e_{i+1} = \overline{e}_i$ for some index i.

However, reduced paths constitute the most important class of paths. We denote by $\mathcal{P}(\Gamma)$ the set of finite reduced edge paths in Γ. Any subset $\mathcal{L}$ of $\mathcal{P}(\Gamma)$ is called a *language over* Γ. Such a language $\mathcal{L}$ is *laminary* if it is (1) non-empty, (2) invariant under orientation-reversion and passage to subpaths, and (3) *bi-extendable*, i.e. every path $\gamma \in \mathcal{L}$ is a non-initial and non-terminal subpath of some strictly longer path $\gamma' \in \mathcal{L}$.

2.2. **Graph maps and train track maps.**

A *graph map* $f : \Gamma \to \Gamma'$ is a map between graphs that sends vertices to vertices and edges to possibly non-reduced edge paths.

For any graph map $f : \Gamma \to \Gamma'$ there is a well defined *transition matrix* (also referred to as *incidence matrix*)

$$M(f) = (m_{e',e})_{e' \in \mathrm{Edges}^{\pm}(\Gamma'),\, e \in \mathrm{Edges}^{\pm}(\Gamma)}\,,$$

where $m_{e',e}$ denotes the number of times that $f(e)$ crosses over e' or over $\overline{e}'$. Both of these occurrences are counted positively, so that $M(f)$ is always a non-negative matrix. One easily verifies

$$M(g \circ f) = M(g) \cdot M(f)$$

for any graph maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma''$.

The reader who is not familiar with unreduced phenomena should be aware of the unpleasant fact that for self-maps $f : \Gamma \to \Gamma$, even if $f(e_i)$ is reduced for every edge e_i, through iterating f, one may well fall upon an edge path $f^t(e_i)$ with $t \geqslant 2$ which turns out to be unreduced. This gives rise to the following important notion, introduced by Bestvina-Handel in [2] and apparently going back to Thurston:

An edge path γ in Γ is *f-legal* if for any integer $t \geqslant 0$ the edge path $f^t(\gamma)$ is reduced. A graph self-map $f : \Gamma \to \Gamma$ is said to have the *train track property*, or to be a *train track map*, if every edge (considered as edge path of length 1) is f-legal. (In the train track literature the notion "f-legal" is usually called simply "legal", but for the purpose of this paper we prefer the more explicit notation.)

A train track map $f : \Gamma \to \Gamma$ is *expanding* if for every edge e of Γ there is an exponent $t \geqslant 1$ such that $f^t(e)$ has length $|f^t(e)| \geqslant 2$.

To any train track map $f : \Gamma \to \Gamma$ there are canonically associated two important languages $\mathcal{L}_\infty(f)$ and $\mathcal{L}_{used}(f)$, defined as follows:

The language $\mathcal{L}_\infty(f)$ consists of all finite edge paths γ that are *infinitely f-legal*: For any $t \geqslant 0$ there is a f-legal path γ_t in Γ such that γ is a subpath of $f^t(\gamma_t)$.

Similarly, $\mathcal{L}_{used}(f)$ is the set of all *f-used* paths, i.e. finite edge paths that are subpaths of some $f^t(e_i)$ for any edge e_i of Γ and any $t \geqslant 1$.

It is easy to see that, if f is expanding, then both languages $\mathcal{L}_\infty(f)$ and $\mathcal{L}_{used}(f)$ are laminary.

Since f is assumed to be a train track map, every edge is f-legal, so that we see directly $\mathcal{L}_{used}(f) \subseteq \mathcal{L}_\infty(f)$. The converse inclusion is in general not true, but for expanding train track maps the difference between the two laminary languages is well understood and not very large (see [15]).

It is important to note that both of these laminary languages are f-invariant:

$$f(\mathcal{L}_{used}(f)) \subseteq \mathcal{L}_{used}(f) \qquad \text{and} \qquad f(\mathcal{L}_\infty(f)) \subseteq \mathcal{L}_\infty(f)$$

2.3. **Marked graphs and representation of free group automorphisms.**

The fundamental group of a graph is always a finitely generated cyclic or non-abelian free group, but for many purposes it is useful to be more specific about this issue: For any integer $N \geqslant 1$ we fix a "model free group" F_N of rank N, and we say that a marking on Γ is an isomorphisms $\theta : \pi_1 \Gamma \xrightarrow{\cong} F_N$. Since we do not want to specify a base point of Γ, marking isomorphisms are only well defined up to composition with inner automorphisms of F_N.

As a consequence, if Γ is a graph equipped with a marking isomorphism θ, then any graph self-map $f : \Gamma \xrightarrow{\simeq} \Gamma$ which is a *homotopy equivalence* defines an outer automorphism φ (i.e. a coset in the group of automorphims of F_N modulo the normal subgroup which consists of all conjugations of F_N by any fixed element):

$$\varphi = \theta f_* \theta^{-1} \in \mathrm{Out}(F_N)\,.$$

In this case one also says that f *represents the automorphism* φ.

If $f : \Gamma \to \Gamma$ is not a homotopy equivalence, then it induces a (possibly non-injective) endomorphism of F_N, but since for such endomorphism the general theory is much less developed than for automorphisms of F_N, in this case we usually refrain from transferring the combinatorial data of the self-map f into an algebraic F_N-setting. This situation will be treated explicitly below in subsection 2.7.

2.4. **Double boundary, laminations and currents.**

If one picks a basis $\mathcal{A}$ for the free group F_N, then every element of F_N is represented by a unique reduced word w in $\mathcal{A} \cup \mathcal{A}^{-1}$ (i.e. $w = x_1 \dots x_q$ with $x_i \in \mathcal{A}$ or $x_i^{-1} \in \mathcal{A}$ and $x_i \neq x_{i+1}^{-1}$ for all indices i). Similarly, the points of the *Gromov boundary* ∂F_N can be represented by right-infinite reduced words $X = x_1 x_2, \dots$, and conversely, each such word defines a point of ∂F_N. We define the *double boundary* of F_N by

$$\partial^2 F_N := \partial F_N \times \partial F_N \smallsetminus \Delta\,,$$

where Δ denotes the diagonal $\{(X, X) \mid X \in \partial F_N\}$.

The boundary ∂F_N comes equipped with a standard "product" topology (indeed, ∂F_N is a Cantor set), and with a left multiplication by elements from F_N. Both structures are naturally inherited by $\partial^2 F_N$, and we define an *algebraic lamination* to be a non-empty subset $L \subseteq \partial^2 F_N$ which is closed, F_N-invariant and invariant under the *flip map* $(X, X') \mapsto (X', X)$.

A *current* μ on F_N is a Borel measure on $\partial^2 F_N$ which is invariant under the F_N-action and the flip map, and which is *finitary*: The measure $\mu(K)$ of any compact set K is finite.

Currents on F_N are much studied (see for instance [8], [12], [13]): The set $\mathrm{Curr}(F_N)$ of such currents is naturally equipped with a topology, a linear structure, and an action of the group $\mathrm{Out}(F_N)$. All three structures are inherited by the quotient space $\mathbb{P}\mathrm{Curr}(F_N)$ which is furthermore compact, though infinitely dimensional. It possesses a canonical "interior" on which the $\mathrm{Out}(F_N)$-action is properly discontinuous, so that it is indeed a valuable analogue for $\mathrm{Out}(F_N)$ of what Teichmüller space is for the mapping class group. This is one of the reason why there is a natural interest in currents which are projective fixed by some $\varphi \in \mathrm{Out}(F_N)$.

2.5. **Universal covering of Γ, laminary languages, Kolmogorov functions.**

Choosing a basis $\mathcal{A}$ for F_N is equivalent to identifying F_N with $\pi_1\Gamma$, where Γ is a 1-vertex graph (called a *rose*), via an identification of $\mathrm{Edges}^+(\Gamma)$ with $\mathcal{A}$. In this case the Gromov boundary ∂F_N is naturally identified with the set $\partial\widetilde{\Gamma}$ of ends of the universal covering $\widetilde{\Gamma}$, which is a simplicial tree. Both, the end-topology as well as the F_N-action as deck transformations on $\widetilde{\Gamma}$ yield precisely the above topology and F_N-action on ∂F_N.

Indeed, the very same statements are true for any graph Γ with marking isomorphism $\pi_1\Gamma \to F_N$. This enables us to translate the above algebraic notions into equivalent combinatorial ones:

For any algebraic lamination $L \subseteq \partial^2 F_N$ and any element $(X, X') \in L$ we consider the biinfinite reduced edge path $\gamma(X, X')$ in $\widetilde{\Gamma}$ which connects the end of $\widetilde{\Gamma}$ given by X to the end given by X'. The language $\mathcal{L} = \mathcal{L}^{\Gamma}(L)$ associated to L is defined as the set of all finite paths γ in Γ that have a lift $\widetilde{\gamma}$ which occurs as subpath of some $\gamma(X, X')$ with $(X, X') \in L$. It is not hard to see that this language is laminary.

Conversely, let $\mathcal{L}$ be any language over Γ, i.e. $\mathcal{L}$ is a subset of the set $\mathcal{P}(\Gamma)$ of finite reduced paths in Γ. If $\mathcal{L}$ is infinite, then it *generates* an algebraic lamination $L = L^{F_N}(\mathcal{L}) \subseteq \partial^2 F_N$, defined as the subset of all $(X, X') \in \partial^2 F_N$ such that all finite subpaths of the geodesic $\gamma(X, X')$ are lifts of paths in $\mathcal{L}$.

To any train track map $f : \Gamma \to \Gamma$ there are canonically associated two important algebraic laminations: the *infinitely f-legal lamination* $L^{F_N}_{\infty}(f) = L^{F_N}(\mathcal{L}_{\infty}(f))$, and the *$f$-used lamination* $L^{F_N}_{used}(f) = L^{F_N}(\mathcal{L}_{used}(f))$ (compare [15]), for the two laminary languages $\mathcal{L}_{\infty}(f)$ and $\mathcal{L}_{used}(f)$ defined above in subsection 2.2.

For any current $\mu \in \mathrm{Curr}(F_N)$ the marking isomorphism $\pi_1\Gamma \cong F_N$ defines canonically a function

$$\mu_{\Gamma} : \mathcal{P}(\Gamma) \to \mathbb{R}_{\geqslant 0}$$

defined on the set $\mathcal{P}(\Gamma)$ through $\mu_{\Gamma}(\gamma) := \mu(C^2_{\widetilde{\gamma}})$, where $\widetilde{\gamma}$ denotes any lift of γ to $\widetilde{\Gamma}$, and the *double cylinder* $C^2_{\widetilde{\gamma}}$ denotes the set of endpoint pairs $(X, Y) \in \partial^2 F_N$ such that $\gamma(X, Y)$ or $\gamma(Y, X)$ contains $\widetilde{\gamma}$ as subpath. Since μ is F_N-invariant, for the definition of $\mu_{\Gamma}(\gamma)$ it doesn't matter which lift $\widetilde{\gamma}$ one considers.

This function μ_{Γ} is a *Kolmogorov function* in that it satisfies for every $\gamma = e_1 e_2 \ldots e_q \in \mathcal{P}(\Gamma)$ the equality

$$\mu_{\Gamma}(\gamma) = \mu_{\Gamma}(\overline{\gamma}) \tag{2.1}$$

and the *Kirchhoff rules*:

$$\mu_{\Gamma}(\gamma) = \sum_{\substack{e_0 \in \mathrm{Edges}^{\pm}(\Gamma) \smallsetminus \{\overline{e}_1\} \\ e_0\gamma \in \mathcal{P}(\Gamma)}} \mu_{\Gamma}(e_0\gamma) = \sum_{\substack{e_{q+1} \in \mathrm{Edges}^{\pm}(\Gamma) \smallsetminus \{\overline{e}_q\} \\ \gamma e_{q+1} \in \mathcal{P}(\Gamma)}} \mu_{\Gamma}(\gamma e_{q+1}) \tag{2.2}$$

The converse holds also: Every Kolmogorov function $\mu_{\Gamma} : \mathcal{P}(\Gamma) \to \mathbb{R}_{\geqslant 0}$ comes from a well defined current $\mu \in \mathrm{Curr}(F_N)$ through the above given definition. The passage back and forth is canonical, so that one has a canonical 1-1 correspondence between currents μ over F_N and Kolmogorov functions μ_{Γ} on the marked graph Γ.

It is not hard to verify that for every current μ on F_N with associated Kolmogorov function μ_{Γ} the support of μ in $\partial^2 F_N$ is precisely the algebraic lamination generated by the laminary language $\mathcal{L}(\mu_{\Gamma}) \subseteq \mathcal{P}(\Gamma)$ given by all reduced paths γ with $\mu_{\Gamma}(\gamma) > 0$.

2.6. **Images of currents under automorphisms represented by graph maps.**
Let $f : \Gamma \to \Gamma'$ be a graph map between marked graphs Γ and Γ', and let $\mathcal{L}(f)$ denote the set of finite reduced paths in Γ that are mapped by f to reduced paths in Γ'. Let $L^{F_N}(f) := L^{F_N}(\mathcal{L}(f)) \subseteq \partial^2 F_N$ be the algebraic lamination generated by $\mathcal{L}(f)$, if $\mathcal{L}(f)$ is infinite, and set $L^{F_N}(f) := \varnothing$ otherwise.

Transferring a current, or rather, the associated Kolmogorov function, from one graph to another via a homotopy equivalence, is a well studied procedure (see [13]). (Recall that

a graph map f is a homotopy equivalence if and only if the induced map f_* on the fundamental groups is an isomorphism.) The issuing formulas, however, are more tricky than one might expect at first sight, because of cancellation phenomena due to the presence of inverses. However, in the following particular situation they specialize to what is well known in symbolic dynamics:

Proposition 2.1. *Let Γ and Γ' be marked graphs, and let $f : \Gamma \to \Gamma'$ be a graph map that realizes via the two markings an outer automorphism φ on F_N (possibly the identity). Assume that the edges of Γ have been subdivided so that the f-preimage of any vertex is a vertex.*

If $\mu \in Curr(F_N)$ has its support contained in $L^{F_N}(f)$, then the corresponding Kolmogorov functions μ_Γ for μ and $(f_*\mu)_{\Gamma'} := \varphi(\mu)_{\Gamma'}$ for $\varphi(\mu)$ satisfy, for any path γ' in $\mathcal{P}(\Gamma')$:

$$(f_*\mu)_{\Gamma'}(\gamma') = \sum_{\{\gamma_i \in \mathcal{L}(f) \,|\, f(\gamma_i) = \gamma'\}} \mu_\Gamma(\gamma_i)$$

□

2.7. **Symbolic dynamics via reduced paths.**

For any graph Γ we denote by $\Sigma(\Gamma)$ the set of $\mathbb{Z}$-parametrized biinfinite reduced edge paths ("biinfinite words") $\gamma = \ldots e_{n-1}e_n e_{n+1} \ldots$ in Γ. The set $\Sigma(\Gamma)$ is naturally provided with a "product" topology, with a shift map S, and with an inversion $\gamma \mapsto \overline{\gamma} = \ldots e'_{n-1}e'_n e'_{n+1} \ldots$ with $e'_n := e_{-n+1}$.

A *symbolic lamination* on Γ is a non-empty subset $L^\Sigma \subseteq \Sigma(\Gamma)$ which is closed, S-invariant, and invariant under inversion. In symbolic dynamics, symbolic laminations are known under the name of *subshift* on the "alphabet" $\text{Edges}^\pm(\Gamma)$, if we treat each pair e_i and $\overline{e}_i$ as distinct unrelated symbols. For more symbolic dynamics terminology see the next subsection.

To any symbolic lamination L^Σ there is canonically associated a language $\mathcal{L}(L^\Sigma) \subseteq \mathcal{P}(\Gamma)$, which consists of all finite subpaths of paths in L^Σ. It is easy to see that the language $\mathcal{L}(L^\Sigma)$ is laminary, for any symbolic lamination L^Σ.

Conversely, given any laminary language $\mathcal{L} \subseteq \mathcal{P}(\Gamma)$, there is a symbolic lamination $L^\Sigma(\mathcal{L}) \subseteq \Sigma(\Gamma)$ associated to it, and the passage back and forth between language and lamination is canonical. This also establishes a similar canonical 1-1 correspondence between algebraic laminations and symbolic laminations, for any marked graph Γ. For more details see the detailed exposition in [7].

Recall from subsections 2.2 and 2.5 that associated to every train track map $f : \Gamma \to \Gamma$ there are natural laminary languages $\mathcal{L}_{used}(f) \subseteq \mathcal{L}_\infty(f)$ (with corresponding algebraic laminations $L^{F_N}_{used}(f) \subseteq L^{F_N}_\infty(f)$). The above set-up gives us directly two corresponding symbolic laminations $L^\Sigma_{used}(f) \subseteq L^\Sigma_\infty(f)$.

An *invariant measure* μ_Σ for Γ is a finite Borel measure on $\Sigma(\Gamma)$ which is invariant under shift and inversion. It defines a Kolmogorov function μ_Γ on the set $\mathcal{P}(\Gamma)$ of all finite reduced edge paths γ in Γ, given by setting $\mu_\Gamma(\gamma) := \mu_\Sigma(C_\gamma)$, where the *cylinder* $C_\gamma \subseteq \Sigma(\Gamma)$ defined by $\gamma = e_1 \ldots e_r$ is the set of all biinfinte reduced paths $\ldots e'_{n-1}e'_n e'_{n+1} \ldots$ which satisfy $e'_1 = e_1, \ldots, e'_r = e_r$.

Conversely, every Kolmogorov function μ_Γ on $\mathcal{P}(\Gamma)$ defines an invariant measure μ_Σ for Γ which satisfies $\mu_\Sigma(C_\gamma) = \mu_\Gamma(\gamma)$, so that the passage back and forth is again canonical. If Γ is marked, we can pass furthermore to $\partial^2 F_N$ in order to obtain from μ_Σ (via μ_Γ) an associated current μ on F_N. Again, the passage back and forth is canonical, see [8].

The support of any invariant measure μ_Σ is a symbolic lamination $L^\Sigma(\mu_\Sigma)$. Similarly, the set of finite paths γ in Γ with $\mu_\Gamma(\gamma) > 0$ is a laminary language $\mathcal{L}(\mu_\Gamma)$, and if μ_Γ is defined by μ_Σ, then $\mathcal{L}(\mu_\Gamma)$ is the laminary language defined by $L^\Sigma(\mu_\Sigma)$.

For any graph map $f : \Gamma_a \to \Gamma_b$ let Γ'_a be the subdivision of Γ_a obtained from pulling back the vertices via f, and let $f' : \Gamma'_a \to \Gamma_b$ be the map induced by f. As before we denote by $\mathcal{L}(f)$ (and similarly for $\mathcal{L}(f')$) the set of finite reduced paths γ in Γ_a for which $f(\gamma)$ is reduced.

We consider the symbolic laminations $L^\Sigma(f) \subseteq \Sigma(\Gamma_a)$ and $L^\Sigma(f') \subseteq \Sigma(\Gamma'_a)$ defined by $\mathcal{L}(f)$ and $\mathcal{L}(f')$ respectively. The canonical passage from Γ_a to Γ'_a via subdivision gives an "identification" between the two symbolic laminations $L^\Sigma(f)$ and $L^\Sigma(f')$. To a geometric group theorists the most natural way to see this passage is to pass by means of a marking through the associated algebraic lamination. In symbolic dynamics or combinatorics this is done typically through a standard recoding procedure.

This identification between $L^\Sigma(f)$ and $L^\Sigma(f')$ allows us to define for every invariant measure μ_Σ with support in $L^\Sigma(f)$ a canonical "subdivision-image" invariant measure μ'_Σ with support in $L^\Sigma(f')$.

Definition 2.2. Let $f : \Gamma_a \to \Gamma_b$ a graph map, and let μ_Σ be an invariant measure for Γ_a with support in $L^\Sigma(f)$. Then there is a well defined *f-image* invariant measure $f_*\mu_\Sigma$ on Γ_b, defined as follows:

Let Γ'_a and $f' : \Gamma'_a \to \Gamma_b$ be obtained from Γ_a and f through subdividing Γ_a at the f-preimage points of the vertices of Γ_a, and let μ'_Σ be the subdivision-image invariant measure canonically defined by μ_Σ. Then the f-image invariant measure $f_*\mu_\Sigma$ is given by the formulas

$$(f_*\mu)_{\Gamma_b}(\gamma) = \sum_{\{\gamma' \in \mathcal{P}(f') \,|\, f'(\gamma') = \gamma\}} \mu'_{\Gamma'_a}(\gamma')$$

for any reduced path $\gamma \in \mathcal{P}(\Gamma_b)$, where $\mu'_{\Gamma'_a}$ is the Kolmogorov function associated to μ'_Σ, and $(f_*\mu)_{\Gamma_b}$ the one associated to $f_*\mu_\Sigma$.

2.8. **Classical symbolic dynamics and substitutions.**

Let $\mathcal{A} = \{a_1, \ldots, a_N\}$ be a finite set, called *alphabet*. We denote by $\mathcal{A}^*$ the free monoid over $\mathcal{A}$. Its neutral element, the empty word, is denoted by $1_\mathcal{A}$. Furthermore, let

$$\Sigma_\mathcal{A} = \{\ldots x_{-1} x_0 x_1 x_2 \ldots \mid x_i \in \mathcal{A}\}$$

be the set of biinfinite words in $\mathcal{A}$, called the *full shift* over $\mathcal{A}$.

For any two "words" $v = y_1 \ldots y_r$ and $w = z_1 \ldots z_s$ in $\mathcal{A}^*$ we define the *cylinder*

$$[v, w] \subseteq \Sigma_\mathcal{A}$$

as the set of all biinfinite words $\ldots x_{-1} x_0 x_1 x_2 \ldots$ in $\mathcal{A}$ which satisfy $x_{-r+1} = y_1, x_{-r+2} = y_2, \ldots, x_0 = y_r$ and $x_1 = z_1, \ldots, x_s = z_s$. The full shift $\Sigma_\mathcal{A}$, being in bijection with the set $\mathcal{A}^\mathbb{Z}$, is naturally equipped with the product topology, where $\mathcal{A}$ is given the discrete topology. The set of cylinders $[v, w]$, for $v, w \in \mathcal{A}^*$, form a basis of this topology. The full shift $\Sigma_\mathcal{A}$ is compact, and indeed it is a Cantor set.

The shift map $S : \Sigma_\mathcal{A} \to \Sigma_\mathcal{A}$ is defined for $x = \ldots x_{-1} x_0 x_1 x_2 \ldots$ by $S(x) = \ldots y_{-1} y_0 y_1 y_2 \ldots$, with $y_n = x_{n+1}$ for all $n \in \mathbb{Z}$. It is bijective and continuous with respect to the above product topology, and hence a homeomorphism.

A *subshift* is a closed subset X of $\Sigma_{\mathcal{A}}$ which is invariant under the shift map S. Let μ be a finite Borel measure supported on a subshift $X \subseteq \Sigma_{\mathcal{A}}$. The measure is called *invariant* if for every measurable set $A \subseteq X$ one has $\mu(S^{-1}(A)) = \mu(A)$. Such a measure μ is said to be *ergodic* if μ can not be written in any non-trivial way as sum $\mu_1 + \mu_2$ of two invariant measures μ_1 and μ_2 (i.e. $\mu_1 \neq 0 \neq \mu_2$ and $\mu_1 \neq \lambda\mu_2$ for any $\lambda \in \mathbb{R}_{>0}$). An invariant measure is called a *probability measure* if $\mu(X) = 1$, which is equivalent to $\sum\limits_{a_i \in \mathcal{A}} \mu([1_{\mathcal{A}}, a_i]) = 1$.

Remark 2.3. It is well known and easy to show that for any invariant measure μ the function

$$\mu_{\mathcal{A}} : \mathcal{A}^* \to \mathbb{R}_{\geqslant 0},\ w \mapsto \mu([1_{\mathcal{A}}, w])$$

satisfies the Kirchhoff rules (2.2). Conversely, every such function determines an invariant measure through the given values on the cylinders.

Definition 2.4. A *substitution* σ is given by a map

$$\mathcal{A} \to \mathcal{A}^*,\ a_i \mapsto \sigma(a_i)$$

A substitution defines both, an endomorphism of $\mathcal{A}^*$, and a continuous map from $\Sigma_{\mathcal{A}}$ to itself which maps $[v, w]$ to $[\sigma(v), \sigma(w)]$. Both of these maps are also denoted by σ, and both are summarized under the name of "substitution".

For any substitution σ we define the associated language $\mathcal{L}_\sigma \subseteq \mathcal{A}^*$ to be the set of factors (in $\mathcal{A}^*$) of the words $\sigma^n(a_i)$, with $n \geqslant 1$ and $a_i \in \mathcal{A}$ (where "factor" is here synonymous to what is called "subword" in combinatorial group theory).

Define the subshift $X_\sigma \subseteq \Sigma_{\mathcal{A}}$ *associated to the substitution* σ as the set of all $x = \ldots x_{k-1}x_kx_{k+1}\cdots \in \Sigma_{\mathcal{A}}$ such that for any integers $m \geqslant n \in \mathbb{Z}$ the word $x_n \ldots x_m$ is an element of $\mathcal{L}_\sigma$.

For any substitution $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ let $m_{i,j}$ be the number of occurrences of the letter a_i in the word $\sigma(a_j)$. The non-negative matrix

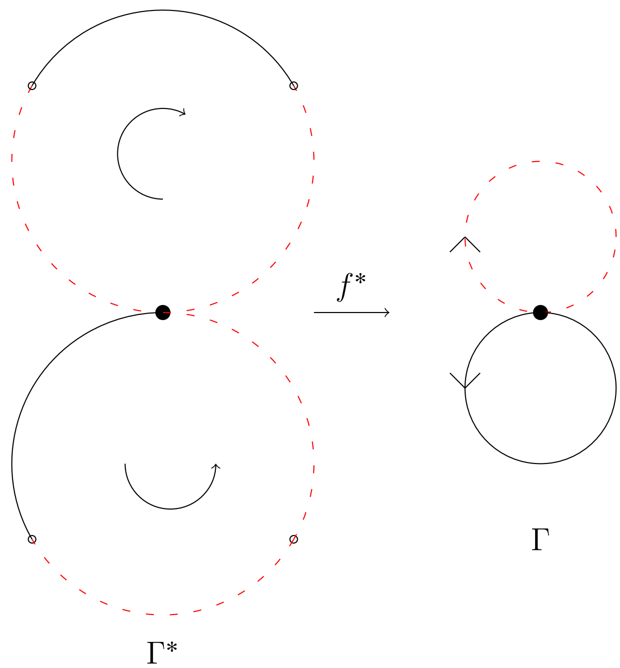
$$M_\sigma := (m_{i,j})_{a_i, a_j \in \mathcal{A}}$$

is called the *incidence matrix* for the substitution σ. The substitution σ is called *primitive* if M_σ is primitive, i.e. there exists an integer k such that every coefficient of M_σ^k is positive.

Remark 2.5. The reader has probably observed already that the classical setting for subshifts and substitutions reviewed above is extremely close to what has been presented in the previous subsection for any graph Γ, in the special case where Γ is a rose $\Gamma_{\mathcal{A}}$ with edge set $\mathrm{Edges}^+(\Gamma_A)$ identified with $\mathcal{A}$ through a bijection $\theta : e_i \mapsto a_i$.

Indeed, a substitution $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ defines canonically a train track map $f_\sigma : \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{A}}$ by setting $f(e_i) := \theta^{-1}(\sigma(a_i))$. The map f_σ has transition matrix $M(f_\sigma) = M_\sigma$. The laminary language $\mathcal{L}_{used}(f_\sigma)$ is equal to $\theta^{-1}(\mathcal{L}_\sigma) \cup \overline{\theta^{-1}(\mathcal{L}_\sigma)}$, where $\overline{\theta^{-1}(\mathcal{L}_\sigma)}$ stands for the set of all $\overline{\gamma}$ with $\gamma \in \theta^{-1}(\mathcal{L}_\sigma)$. As a consequence one obtains the symbolic lamination $L^{\Sigma}_{used}(f)$ as union $\theta^{-1}(X_\sigma) \cup \overline{\theta^{-1}(X_\sigma)}$ from the subshift X_σ.

Furthermore, an invariant measure μ with support on X_σ translates directly into an invariant measure μ_Σ on $\Sigma(\Gamma_{\mathcal{A}})$: From μ we pass over to the associated function $\mu_{\mathcal{A}}$ and make it into a Kolmogorov function μ_Γ for $\Gamma_{\mathcal{A}}$, through setting $\mu_\Gamma(\gamma) = \mu_{\mathcal{A}}(\theta(\gamma))$ if γ uses only edges from $\mathrm{Edges}^+(\Gamma_{\mathcal{A}})$, setting $\mu_\Gamma(\gamma) = \mu_{\mathcal{A}}(\theta(\overline{\gamma}))$ if γ uses only edges from $\mathrm{Edges}^\pm(\Gamma_{\mathcal{A}}) \smallsetminus \mathrm{Edges}^+(\Gamma_{\mathcal{A}})$, and through defining $\mu_\Gamma(\gamma) = 0$ otherwise. The invariant measure μ_Σ is then given canonically as described above by μ_Γ.



Long edge dialect

FIGURE 1.

3. GRAPH MAPS IN SEVERAL DIFFERENT DIALECTS

Convention 3.1. We recall the following conventions, see section 2:

(1) In this paper all graphs are finite, connected, and without vertices of valence 1, but possibly with vertices of valence 2.

(2) A *graph map* $f : \Gamma' \to \Gamma$ is a map between graphs Γ' and Γ which maps vertices to vertices and edges to edge paths.

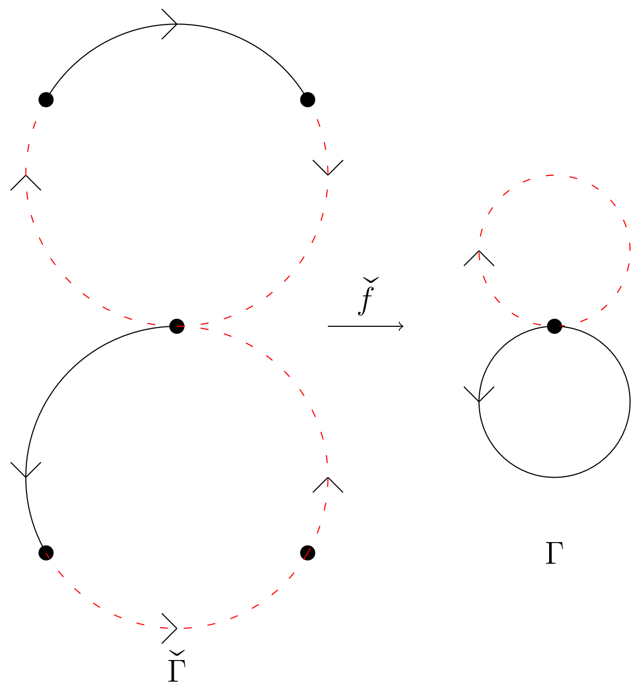
Please note that a priori, the image edge path $f(e)$ in Γ of an edge e of Γ' may not be reduced. It could also be a trivial edge path, i.e. e is contracted by f to a single vertex.

(3) For any edge path γ in Γ we denote by $|\gamma|$ the *combinatorial length* (or simply *length*) of γ, by which we mean the number of edges traversed by γ.

We will now define three different "dialects", in which graphs and graph maps can occur, as well as the formal transition between them. This will be done below with all technical details, since it is the base for what comes in the subsequent sections. However, in a first approach the reader may prefer to only glance quickly through the rest of this section.

Definition 3.2. (1) A graph Γ^* is given in *long-edge dialect* if Γ^* has only *intrinsic* vertices, i.e. vertices of valence $\geqslant 3$. Edges of such a graph are called *long* edges, and we usually denote them by e^* or e_i^*.

A graph map $f^* : \Gamma^* \to \Gamma$ is in *long-edge dialect* if the graph Γ^* is in long-edge dialect, and if f^* *has no contracted edges*, i.e. no edge of Γ^* is mapped by f^* to a trivial edge edge path (i.e. to a single vertex).



Short edge dialect

FIGURE 2.

(2) A graph map $\check{f} : \check{\Gamma} \to \Gamma$ is in *short-edge dialect* if for every edge e of $\check{\Gamma}$ the image path $\check{f}(e)$ has length 1, or in other words: $\check{f}$ maps every edge to a single edge. Such edges are called *f-short*, or simply *short*.

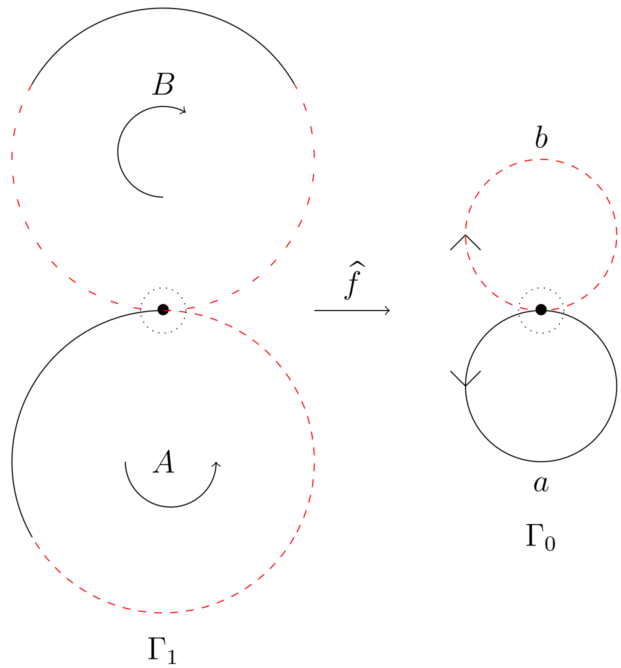
Remark 3.3. The "translation" of any graph Γ', or of graph map $f : \Gamma' \to \Gamma$ without contracted edges, into long-edge dialect is simply given by erasing all valence two vertices from Γ'. We formalize this transition by calling the resulting graph $\mathrm{Long}(\Gamma')$ and the resulting map $\mathrm{Long}(f)$.

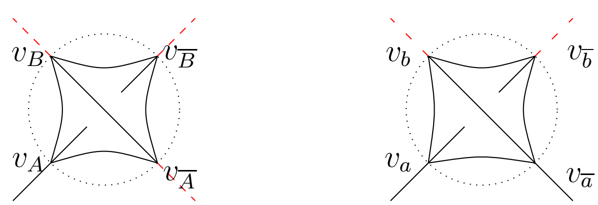
Similarly, the translation into short-edge dialect is given by introducing new valence 2 vertices in Γ' for every f-preimage point of a vertex of Γ (unless, of course, the preimage point is already a vertex of Γ'). Again, we formalize this transition by calling the resulting graph $\mathrm{Short}(\Gamma')$ and the resulting map $\mathrm{Short}(f)$.

The reader verifies directly the following equalities, for any graph map $f : \Gamma' \to \Gamma$ without contracted edges:

$$\mathrm{Long}(\Gamma') = \mathrm{Long}(\mathrm{Long}(\Gamma')) = \mathrm{Long}(\mathrm{Short}(\Gamma'))$$
$$\mathrm{Long}(f) = \mathrm{Long}(\mathrm{Long}(f)) = \mathrm{Long}(\mathrm{Short}(f))$$
$$\mathrm{Short}(\Gamma') = \mathrm{Short}(\mathrm{Short}(\Gamma')) = \mathrm{Short}(\mathrm{Long}(\Gamma'))$$
$$\mathrm{Short}(f) = \mathrm{Short}(\mathrm{Short}(f)) = \mathrm{Short}(\mathrm{Long}(f))$$

Definition 3.4. (1) A graph $\widehat{\Gamma}$ is given in *blow-up dialect* if the following conditions are satisfied:





Blow-up dialect

FIGURE 3.

(a) The vertices of $\widehat{\Gamma}$ are partitioned into classes:

$$V(\widehat{\Gamma}) = V_1 \mathbin{\dot{\cup}} \dots \mathbin{\dot{\cup}} V_q$$

Here $\dot{\cup}$ denotes the disjoint union.

(b) The edges of $\widehat{\Gamma}$ are partitioned into classes:

$$\text{Edges}^{\pm}(\widehat{\Gamma}) = \widehat{E}^{\pm} \mathbin{\dot{\cup}} \mathcal{E}_1^{\pm} \mathbin{\dot{\cup}} \dots \mathbin{\dot{\cup}} \mathcal{E}_q^{\pm}$$

Occasionally we will specify this notation to $\widehat{E}^{\pm}(\widehat{\Gamma}) := \widehat{E}^{\pm}$ and $\mathcal{E}^{\pm}(\widehat{\Gamma}) := \mathcal{E}_1^{\pm} \cup \ldots \cup \mathcal{E}_q^{\pm}$.

(c) For every $k = 1, \ldots, q$ the edges ε_j from $\mathcal{E}_k^{\pm}$ (called *local edges*) form a complete graph (called *local vertex graph*) over the vertex set V_k.

(d) Every vertex is the endpoint of precisely one edge $\widehat{e}_i$ from $\widehat{E}^{\pm}$.

(2) A graph map $\widehat{f} : \widehat{\Gamma}' \to \widehat{\Gamma}$ is given in *blow-up dialect* if both, $\widehat{\Gamma}$ and $\widehat{\Gamma}'$ are in blow-up dialect, and if the map $\widehat{f}$ maps every local vertex graph of $\widehat{\Gamma}'$ to a local vertex graph of $\widehat{\Gamma}$.

Here every local edge ε'_j of $\widehat{\Gamma}'$ is either mapped to a single local edge $\varepsilon_k = \widehat{f}(\varepsilon'_j)$ of $\widehat{\Gamma}$, or else ε'_j is contracted by $\widehat{f}$ to a vertex. In the first case the local edge ε'_j will be termed *legal*, while in the second case we call it *illegal*.

We also require that for every non-local edge $\widehat{e}' \in \widehat{E}^{\pm}$ of $\widehat{\Gamma}'$ the image edge path $\widehat{f}(\widehat{e}')$ does not have a local edge as initial or as terminal edge, and that $\widehat{f}(\widehat{e}')$ never traverses two consecutive local edges.

Remark 3.5. (1) Let $\widehat{\Gamma}$ be a graph in blow-up dialect. The graph Γ obtained from $\widehat{\Gamma}$ by contracting all local edges of $\widehat{\Gamma}$ (and hence identifying, for each $k = 1, \ldots q$, all vertices in V_k to define a single quotient vertex $\mathcal{V}_k$), is said to be obtained *by contraction*. We denote this by:

$$\Gamma = \mathrm{Contr}(\widehat{\Gamma})$$

(2) Let $\widehat{f} : \widehat{\Gamma}' \to \widehat{\Gamma}$ be a graph map in blow-up dialect. We say that the map $f : \Gamma' \to \Gamma$ is obtained from $\widehat{f}$ *by contraction* if we have $\Gamma = \mathrm{Contr}(\widehat{\Gamma})$ and $\Gamma' = \mathrm{Contr}(\widehat{\Gamma}')$, and f is the map induced by $\widehat{f}$ on the two quotient graphs. In this case we write:

$$f = \mathrm{Contr}(\widehat{f})$$

We now want to describe the converse "translation". For this purpose we first define a *blow-up* procedure at a vertex v of the graph Γ: Let $E(v)$ be the set of oriented edges e which have v as initial vertex (i.e. if some edge e has v as initial and also as terminal vertex, then both, e and $\overline{e}$ belong to $E(v)$). We define a *local vertex graph* $\Gamma(v)$, which has a vertex v_e for each $e \in E(v)$ and is the full graph over this *local vertex set* $\{v_e \mid e \in E(v)\}$. The edges of such as local graph $\Gamma(v)$ are called *local edges* and will be denoted by ε or ε_k.

Definition 3.6. (1) For any graph Γ the associated *blow-up graph* $\widehat{\Gamma}$ is defined as the union of the local vertex graphs $\Gamma(v)$, for any vertex v of Γ, together with an edge $\widehat{e}$ for every edge e of Γ: if e has initial vertex v_1 and terminal vertex v_2, then the initial vertex of $\widehat{e}$ is the *local vertex* v_e of $\Gamma(v_1)$, and the terminal vertex of $\widehat{e}$ is the local vertex $v_{\overline{e}}$ of $\Gamma(v_2)$.

One verifies easily that the conditions (a) - (d) of Definition 3.4 (1) are satisfied. We write:

$$\widehat{\Gamma} = \text{Blow-up}(\Gamma)$$

(2) Given a graph map $f : \Gamma' \to \Gamma$ which maps edges to reduced non-trivial edge paths, we define the associated *blow-up map* $\widehat{f} : \widehat{\Gamma}' \to \widehat{\Gamma}$ by passing to the blow-up graphs $\widehat{\Gamma} :=$ Blow-up(Γ) and $\widehat{\Gamma}' :=$ Blow-up(Γ'). For any edge e' of Γ' with $f(e') = e_1 e_2 \ldots e_t$ we define $\widehat{f}(\widehat{e}') := \widehat{e}_1 \varepsilon_1 \widehat{e}_2 \varepsilon_2 \ldots \varepsilon_{t-1} \widehat{e}_t$, where ε_k is the local edge that connects the terminal vertex $v_{\overline{e}_k}$ of $\widehat{e}_k$ to the initial vertex $v_{e_{k+1}}$ of $\widehat{e}_{k+1}$. (Note that such a local edge must exist, since the terminal vertex of e_k agrees with the initial vertex of e_{k+1} in Γ, and since from the assumption that $f(e')$ is reduced it follows that $v_{\overline{e}_k} \neq v_{e_{k+1}}$.)

For any local edge ε'_j of $\widehat{\Gamma}'$ which connects a vertex v_1 to a vertex v_2, the image $\widehat{f}(\varepsilon'_j)$ is either defined to be the local edge connecting $\widehat{f}(v_1)$ to $\widehat{f}(v_2)$, in case that these two vertices are distinct, or else $\widehat{f}(\varepsilon'_j)$ is contracted to the single vertex $\widehat{f}(v_1) = \widehat{f}(v_2)$.

Again, one sees directly that the map $\widehat{f}$ is in blow-up dialect, as set up above in Definition 3.4 (2). We write:

$$\widehat{f} = \text{Blow-up}(f)$$

Remark 3.7. (1) The reader verifies directly from the definitions the following equalities, for any graph Γ, or for any graph map $f : \Gamma' \to \Gamma$ which maps edges to reduced non-trivial edge paths:

$$\Gamma = \text{Contr}(\text{Blow-up}(\Gamma))$$
$$f = \text{Contr}(\text{Blow-up}(f))$$

Similarly, for any graph $\widehat{\Gamma}$ and any graph map $f : \widehat{\Gamma}' \to \widehat{\Gamma}$ in blow-up dialect we have:

$$\widehat{\Gamma} = \text{Blow-up}(\text{Contr}(\widehat{\Gamma}))$$
$$\widehat{f} = \text{Blow-up}(\text{Contr}(\widehat{f}))$$

(2) In the next sections the blow-up dialect will almost always be used in combination with the short edge dialect, i.e. we will consider, for a given graph map $f : \Gamma' \to \Gamma$ the combined translations $\text{Blow-up}(\text{Short}(f)) : \text{Blow-up}(\text{Short}(\Gamma')) \to \text{Blow-up}((\Gamma))$.

However, there can also be instances where one uses the blow-up dialect in combination with the long-edge dialect, i.e. one works with the maps $\text{Blow-up}(\text{Long}(f)) : \text{Blow-up}(\text{Long}(\Gamma')) \to \text{Blow-up}((\Gamma))$.

Convention 3.8. In the subsequent sections we will occasionally pass in an informal way from one dialect to the other. In this case we use the following convention, for any graph map $f : \Gamma' \to \Gamma$, and any edge path γ in Γ':

The path γ will not change name if we pass to long-edge or to short-edge dialect: Indeed, γ stays topologically the same, as simply valence 2 vertices will be added or removed. In long-edge dialect it could hence be that γ is not any more an edge path in the classical sense, but starts and finishes with a "partial edge" (or γ may also be entirely contained in a single long edge).

In the case where we pass to the blow-up dialect, the name γ still stays, but in addition we impose that in the blow-up graph $\widehat{\Gamma}'$ the corresponding path γ never starts or ends with a local edge, and never passes over two consecutive local edges.

It is a direct consequence of the above conventions that changing back and forth dialects will not change γ if after several changes one ends up in the same dialect as started out with. Here we need to assume, if we start out in blow-up dialect, that γ does not start or end with a local edge, and does never pass over two consecutive local edges.

4. Graph towers

Let F_N be a non-abelian free group of finite rank $N \geqslant 2$. Let Γ be a graph, provided with a marking isomorphism $\theta : \pi_1 \Gamma \xrightarrow{\cong} F_N$ (see section 2).

The purpose of this section is to introduce the main tool of this paper, called "graph towers". We will first define them without reference to any of the three dialects introduced in section 3. We comment below about the translation into these dialects.

Definition 4.1. A *graph tower* $\overleftarrow{\Gamma}$ is given by an infinite family $(\Gamma_n)_{n \in \mathbb{N} \cup \{0\}}$ of finite connected *level graphs* Γ_n, and an infinite family $\overleftarrow{f} = (f_{m,n})_{0 \leqslant m \leqslant n}$ of graph maps $f_{m,n} : \Gamma_n \to \Gamma_m$ with the following properties:

(a) $f_{m,n}$ maps vertices to vertices.
(b) $f_{m,n}$ maps edges to reduced non-trivial edge paths.
(c) The family $\overleftarrow{f}$ is *compatible*: one has $f_{k,m} \circ f_{m,n} = f_{k,n}$ for all integers $n \geqslant m \geqslant k \geqslant 0$. In particular we require $f_{n,n} = id_{\Gamma_n}$ for all $n \geqslant 0$.

For simplicity we will use the abbreviations $f_n := f_{0,n}$ for all $n \geqslant 0$.

Furthermore, if Γ_0 is identified with a marked graph Γ (see subsection 2.3), we say that $\overleftarrow{\Gamma}$ is a graph tower *over the marked graph* Γ, or simply that $\overleftarrow{\Gamma}$ is a *marked graph tower*.

Remark 4.2. (1) Any graph tower $\overleftarrow{\Gamma} = ((\Gamma_n)_{n \in \mathbb{N} \cup \{0\}}, (f_{m,n})_{0 \leqslant m \leqslant n})$ can be translated canonically into

- a *long-edge graph tower* $((\Gamma^*_n)_{n \in \mathbb{N} \cup \{0\}}, (f^*_{m,n})_{0 \leqslant m \leqslant n})$,
- a *short-edge graph tower* $((\check{\Gamma}_n)_{n \in \mathbb{N} \cup \{0\}}, (\check{f}_{m,n})_{0 \leqslant m \leqslant n})$, or
- a *blow-up graph tower* $((\widehat{\Gamma}_n)_{n \in \mathbb{N} \cup \{0\}}, (\widehat{f}_{m,n})_{0 \leqslant m \leqslant n})$.

This translation is always done "from the bottom up", always following carefully the instructions explained in section 3: One first translates Γ_0, then Γ_1 together with f_1, then Γ_2 together with f_2 and $f_{1,2}$, and so on.

(2) As a consequence, we note for any level graph Γ_n that, in the process of translating Γ_n into short-edge dialect $\check{\Gamma}_n$ through subdivision of the edges according to any of the maps $f_{m,n}$ (see Remark 3.3), the set of newly introduced valence 2 vertices is independent of the choice of m, since any of the level graphs Γ_m with $m \leqslant n-1$ has (in the procedure "from the bottom up") already been translated into short-edge dialect.

(3) Similarly, we note that if in the blow-up dialect any level graph $\widehat{\Gamma}_n$ has a local edge ε_k which is *illegal*, by which we mean "illegal with respect to the map $\widehat{f}_n$" (see Definition 3.4 (2)), then for any level graph $\widehat{\Gamma}_m$ of lower level $m \leqslant n$ the image $\widehat{f}_{m,n}(\varepsilon_k)$ is either degenerated to a single vertex, or else $\widehat{f}_{m,n}(\varepsilon_k)$ is a local edge which then must also be illegal (i.e. illegal with respect to $\widehat{f}_m$).

Definition 4.3. We say that the graph tower $\overleftarrow{\Gamma}$, given by a family $\overleftarrow{f}$ of graph maps as in Definition 4.1, is *expanding* if, when considering the long edges e^*_i obtained from deleting the non-intrinsic vertices of the level graphs Γ_n (i.e. by passing over to level graphs Γ^*_n through translation into long-edge dialect) the *minimal long edge length*

$$\text{minlength}_{\overleftarrow{\Gamma}}(n) := \min_{e^*_i \in \text{Edges}^{\pm}(\Gamma^*_n)} |f^*_n(e^*_i)|$$

satisfies

$$\lim_{n \to \infty} (\text{minlength}_{\overleftarrow{\Gamma}}(n)) \to \infty\,.$$

An edge path γ in a level graph Γ_n of a graph tower $\overleftarrow{\Gamma}$ as above is called *legal* if its image $f_n(\gamma)$ is reduced. This is equivalent to stating, where we use the translation into the blow-up dialect, that γ only crosses over local edges ε_k that are legal, as has been specified in Remark 4.2 (3).

It follows directly that γ is legal if and only if all paths $f_{m,n}(\gamma)$ (with $n \geqslant m \geqslant 0$) are legal. We denote by $\mathcal{P}_{legal}(\Gamma_n)$ the set of all finite legal paths in Γ_n.

Definition 4.4. Every expanding graph tower $\overleftarrow{\Gamma} = ((\Gamma_n)_{n \in \mathbb{N} \cup \{0\}}, (f_{m,n})_{0 \leqslant m \leqslant n})$ defines a set $\mathcal{P}_{legal}(\overleftarrow{\Gamma})$ of *infinitely legal* edge paths in Γ_0, given by:

$$\mathcal{P}_{legal}(\overleftarrow{\Gamma}) = \bigcap_{n \geqslant 0} f_n(\mathcal{P}_{legal}(\Gamma_n))$$

We denote by

$$L^{\Sigma}_{legal}(\overleftarrow{\Gamma}) := L^{\Sigma}(\mathcal{P}_{legal}(\overleftarrow{\Gamma})) \subseteq \Sigma(\Gamma_0)$$

the *infinitely legal* symbolic lamination generated by the set $\mathcal{P}_{legal}(\overleftarrow{\Gamma})$ (see subsection 2.7). If $\overleftarrow{\Gamma}$ is marked, then $\mathcal{P}_{legal}(\overleftarrow{\Gamma})$ generates an algebraic lamination for F_N (see subsection 2.4), called the *infinitely legal tower lamination*:

$$L^{F_N}_{legal}(\overleftarrow{\Gamma}) := L^{F_N}(\mathcal{P}_{legal}(\overleftarrow{\Gamma})) \subseteq \partial^2 F_N$$

Remark 4.5. Every expanding graph tower $\overleftarrow{\Gamma}$ as above also defines an infinite set $\mathcal{P}_{used}(\overleftarrow{\Gamma})$ of *used* edge paths in Γ, given by

$$\mathcal{P}_{used}(\overleftarrow{\Gamma}) = \{f_n(e^*) \mid e^* \in \text{Edges}(\Gamma^*_n), n \geqslant 0\},$$

which generates the *used* symbolic lamination

$$L^{\Sigma}_{used}(\overleftarrow{\Gamma}) := L^{\Sigma}(\mathcal{P}_{used}(\overleftarrow{\Gamma})) \subseteq \Sigma(\Gamma).$$

If $\overleftarrow{\Gamma}$ is marked, then $\mathcal{P}_{used}(\overleftarrow{\Gamma})$ generates an algebraic lamination for F_N, called the *used tower lamination*:

$$L^{F_N}_{used}(\overleftarrow{\Gamma}) := L^{F_N}(\mathcal{P}_{used}(\overleftarrow{\Gamma})) \subseteq \partial^2 F_N$$

It is easy to see that this is a smaller lamination than the above defined infinitely legal tower lamination, but in general the difference is not very large (often indeed consisting of finitely many F_N-orbits):

$$L^{F_N}_{used}(\overleftarrow{\Gamma}) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma})$$

5. Weights and Currents

Convention 5.1. In this section we assume that any graph tower $\overleftarrow{\Gamma} = ((\Gamma_n)_{n \in \mathbb{N} \cup \{0\}}, (f_{m,n})_{0 \leqslant m \leqslant n})$ is given in short-edge dialect (see Definition 3.3 and Remark 4.2). In other words: we have $\Gamma_n = \check{\Gamma}_n$ and $f_{m,n} = \check{f}_{m,n}$ for all $n \geqslant m \geqslant 0$.

Definition 5.2. (1) Let Γ be a graph and let $\widehat{\Gamma}$ be the associated blow-up graph as in Definition 3.6 (1). We first define a *weight function* $\widehat{\omega}$ on $\widehat{\Gamma}$: This is a non-negative function

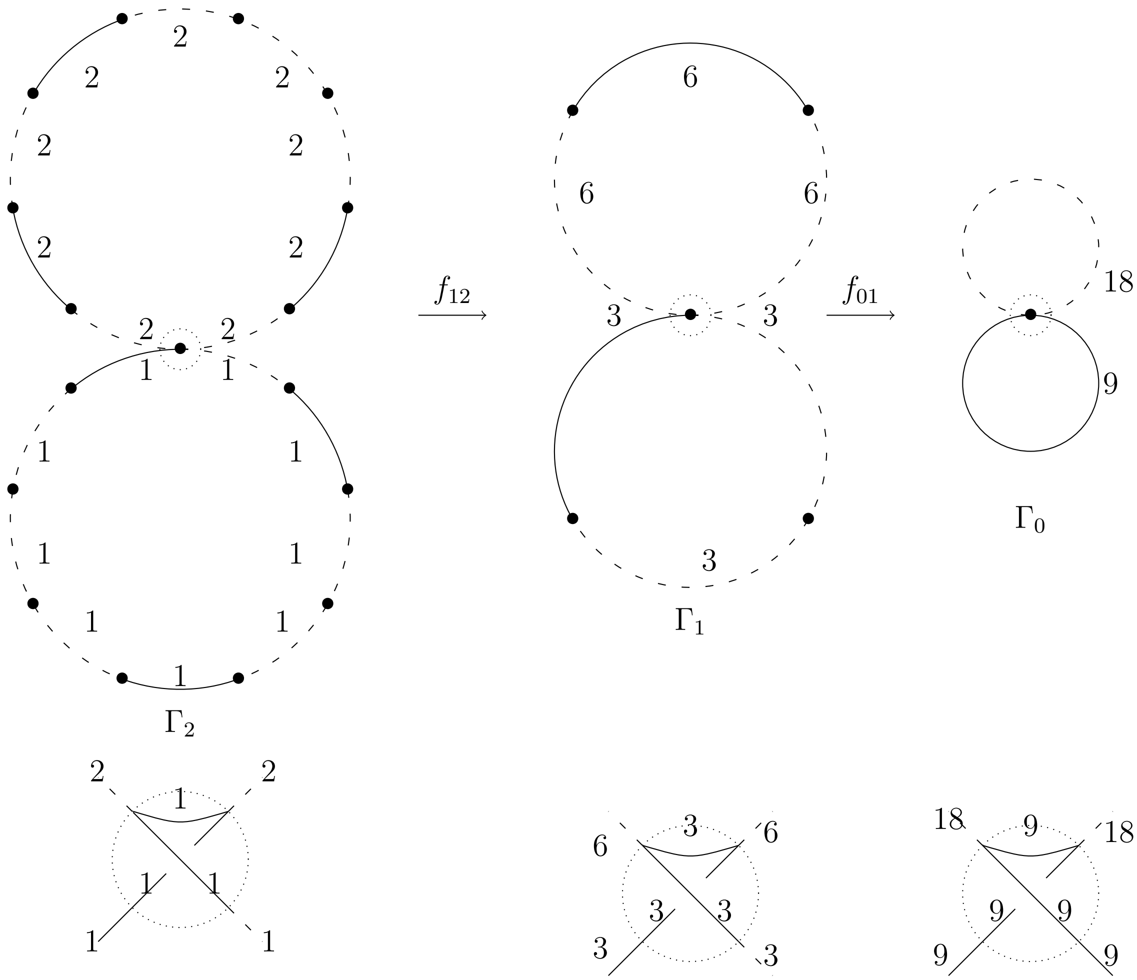
$$\widehat{\omega} : \text{Edges}^{\pm}(\widehat{\Gamma}) \to \mathbb{R}_{\geqslant 0} \quad \text{with} \quad \widehat{\omega}(\overline{\widehat{e}}) = \widehat{\omega}(\widehat{e}) \quad \text{for all} \quad \widehat{e} \in \text{Edges}(\widehat{\Gamma})$$

which in addition satisfies the following *switch conditions*: For every non-local edge $\widehat{e}$ of $\widehat{\Gamma}$ which has the local vertex $v_{\widehat{e}}$ as initial vertex, one has

$$\widehat{\omega}(\widehat{e}) = \sum_{\varepsilon_k \in E(v_{\widehat{e}})} \widehat{\omega}(\varepsilon_k), \tag{5.1}$$

where $E(v_{\widehat{e}})$ is the set of all local edges ε_k with initial vertex $v_{\widehat{e}}$.

A *weight function* ω on Γ is a function that is induced by some weight function $\widehat{\omega}$ on the associated blow-up graph $\widehat{\Gamma}$, i.e. for every edge e of Γ and the associated non-local edge $\widehat{e}$ of $\widehat{\Gamma}$ one has $\omega(e) = \widehat{\omega}(\widehat{e})$.



Levels $2, 1$ and 0 of weighted graph tower

Values of the associated Kolmogorov function μ_Γ read off from the weighted graph tower (not listing the inverses, since $\mu_\Gamma(w^{-1}) = \mu_\Gamma(w)$, not listing any word w with at least one positive and one negative exponent, as they all satisfy $\mu_\Gamma(w) = 0$, and not listing any positive w with $|w| \geqslant 4$ and $\mu_\Gamma(w) = 0$). Listing for example $\mu_\Gamma(bababb) = 1 + 2$ means that $bababb$ occurs once with weight 1 and once with weight 2.:

- $\mu_\Gamma(a) = 9$, $\mu_\Gamma(b) = 18$
- $\mu_\Gamma(aa) = 0$, $\mu_\Gamma(ab) = 9$, $\mu_\Gamma(ba) = 9$, $\mu_\Gamma(bb) = 9$
- $\mu_\Gamma(aaa) = 0$, $\mu_\Gamma(aab) = 0$, $\mu_\Gamma(aba) = 3$, $\mu_\Gamma(abb) = 3 + 3$, $\mu_\Gamma(baa) = 0$, $\mu_\Gamma(bab) = 6 + 3$, $\mu_\Gamma(bba) = 3 + 3$, $\mu_\Gamma(bbb) = 3$
- $\mu_\Gamma(abba) = 3$, $\mu_\Gamma(abbb) = 3$, $\mu_\Gamma(babb) = 3 + 3$, $\mu_\Gamma(baba) = 3$
- $\mu_\Gamma(abbab) = 3$, $\mu_\Gamma(ababb) = 3$, $\mu_\Gamma(bbaba) = 3$, $\mu_\Gamma(bbabb) = 3$, $\mu_\Gamma(bbbaa) = 3$
- $\mu_\Gamma(ababbb) = 1 + 2$, $\mu_\Gamma(abbaba) = 1 + 1$, $\mu_\Gamma(abbabb) = 1$, $\mu_\Gamma(abbbab) = 1 + 2$, $\mu_\Gamma(bababb) = 1 + 2$, $\mu_\Gamma(babbab) = 1 + 1 + 1$, $\mu_\Gamma(babbba) = 1 + 2$, $\mu_\Gamma(bbabab) = 1 + 1 + 1$, $\mu_\Gamma(bbabba) = 1 + 1 + 1$, $\mu_\Gamma(bbbaba) = 1$, $\mu_\Gamma(bbbabb) = 1 + 1$

FIGURE 4.

(2) Let $\overleftarrow{\Gamma} = ((\Gamma_n)_{n\in\mathbb{N}\cup\{0\}}, (f_{m,n})_{0\leqslant m\leqslant n})$ be a graph tower as in Convention 5.1, and let $((\widehat{\Gamma}_n)_{n\in\mathbb{N}\cup\{0\}}, (\widehat{f}_{m,n})_{0\leqslant m\leqslant n})$ be the associated blow-up graph tower (see Remark 4.2).

A *tower of weight functions* (or simply a *weight tower*) $\overleftarrow{\omega}$ on $\overleftarrow{\Gamma} = ((\Gamma_n)_{n\in\mathbb{N}\cup\{0\}}, (f_{m,n})_{0\leqslant m\leqslant n})$ is a family of weight functions $\omega_n : \mathrm{Edges}^{\pm}(\Gamma_n) \to \mathbb{R}_{\geqslant 0}$ which is induced by a family of weight functions $\widehat{\omega}_n : \mathrm{Edges}^{\pm}(\widehat{\Gamma}_n) \to \mathbb{R}_{\geqslant 0}$ as in part (1) above. The functions ω_n satisfy for all integers $n \geqslant m \geqslant 0$ and any edge $e \in \mathrm{Edges}(\Gamma_m)$ the following *compatibility condition*:

$$\omega_m(e) = \sum_{\{e_i\in\mathrm{Edges}^{\pm}(\Gamma_n)\,|\, f_{m,n}(e_i)=e\}} \omega_n(e_i) \tag{5.2}$$

Similarly, for any local edge ε of $\widehat{\Gamma}_m$ one has:

$$\widehat{\omega}_m(\varepsilon) = \sum_{\{\varepsilon_k\in\mathcal{E}^{\pm}(\widehat{\Gamma}_n)\,|\, \widehat{f}_{m,n}(\varepsilon_k)=\varepsilon\}} \widehat{\omega}_n(\varepsilon_k) \tag{5.3}$$

Remark 5.3. (1) From the switch conditions (5.1) and the compatibility conditions (5.2) and (5.3) together it follows directly (see Remark 4.2 (3)) that every illegal local edge ε_i at any vertex of any level graph Γ_n must have weight $\widehat{\omega}_n(\varepsilon_i) = 0$. Indeed, any such ε_i is mapped by some $f_{m,n}$ to a single local vertex, and as a result, if the compatibility conditions for $f_{m,n}$ are valid, then $\widehat{\omega}_n(\varepsilon_i) \neq 0$ would imply that the switch conditions for $\widehat{\omega}_m$ at this local vertex fail, assuming that for $\widehat{\omega}_n$ they are valid.

(2) We observe that any weight function ω_n on a level graph Γ_n induces a weight function ω_n^* on the long-edge dialect level graph Γ_n^* associated to Γ_n, with the property $\omega_n^*(e^*) = \omega_n(e_i)$ for any long edge e^* of Γ_n^*, and any edge e_i of Γ_n which arises from subdividing e^*. This is a consequence of the fact that at any subdivision vertex v_i on e^*, say equal to the terminal vertex of e_{i-1} and the initial vertex of e_i, the local vertex graph $\Gamma(v_i)$ consists only of a single local edge ε_i, so that the switch conditions give:

$$\omega_n(e_{i-1}) = \widehat{\omega}_n(\widehat{e}_{i-1}) = \widehat{\omega}_n(\varepsilon_i) = \widehat{\omega}_n(\overline{\varepsilon}_i) = \widehat{\omega}_n(\overline{\widehat{e}_i}) = \widehat{\omega}_n(\widehat{e}_i) = \omega_n(e_i)$$

As a consequence, we see that in terms of weight functions the local edges at valence 2 vertices of Γ_n do not really play any important role.

However, one should keep in mind that, in the compatibility condition (5.3) for the local edges, for the sum on the right hand side, the summation has to be taken over *all* local edges ε_k that are mapped by $\widehat{f}_{m,n}$ to ε, which includes also the local edge of the local vertex graph of any blown-up valence 2 vertex.

(3) For simplicity, since no confusion is to be feared, we will from now on drop the hat of $\widehat{\omega}_m$ and denote the weight of any local edge ε of any level graph Γ_m simply by $\omega_m(\varepsilon)$.

As a consequence of part (2) of Remark 5.3 we observe for any edge path $\gamma = e_1 e_2 \ldots e_q$ in Γ_n that, if γ is entirely contained in some edge e^* from the associated long-edge dialect graph Γ_n^*, all edges e_i traversed by γ have the same weight. Thus setting

$$\omega_n(\gamma) := \omega_n(e_i)$$

for any of the traversed edges e_i gives a well defined *weight of the path* γ.

On the other hand, if γ traverses any intrinsic vertex v of Γ_n, i.e. a vertex which is inherited from a vertex of Γ_n^*, then the local edge ε traversed by γ at v and the two edges e

and e' of Γ_n which are adjacent to ε on γ satisfy

$$\omega_n(\varepsilon) \leqslant \omega_n(e), \quad \text{and} \quad \omega_n(\varepsilon) \leqslant \omega_n(e')\,,$$

and these inequalities may well be strict. For such γ, if ε is the only intrinsic local edge traversed by γ, we set:

$$\omega_n(\gamma) := \omega_n(\varepsilon)$$

Proposition 5.4. *Let* $\overleftarrow{\Gamma} = ((\Gamma_n)_{n\in\mathbb{N}\cup\{0\}}, (f_{m,n})_{0\leqslant m\leqslant n})$ *be an expanding graph tower.*

(1) Every tower of weight functions $\overleftarrow{\varpi} = (\omega_n : \Gamma_n \to \mathbb{R}_{\geqslant 0})_{n\in\mathbb{N}\cup\{0\}}$ *on* $\overleftarrow{\Gamma}$ *defines an invariant measure* $\mu_\Sigma^{\overleftarrow{\varpi}}$ *on the infinitely legal symbolic lamination* $L^\Sigma_{legal}(\overleftarrow{\Gamma})$. *If* $\overleftarrow{\Gamma}$ *is marked, then* $\overleftarrow{\varpi}$ *defines a current* $\mu^{\overleftarrow{\varpi}}$ *over* F_N*, and its support satisfies:*

$$Supp(\mu^{\overleftarrow{\varpi}}) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma})$$

(2) More precisely, for every path γ *in* $\Gamma := \Gamma_0$ *we obtain the value of the Kolmogorov function* $\mu_\Gamma^{\overleftarrow{\varpi}}$ *on* γ *by considering any level graph* Γ_n *with* $\text{minlength}_{\overleftarrow{\Gamma}}(n) \geqslant |\gamma|$ *and setting:*

$$\mu_\Gamma^{\overleftarrow{\varpi}}(\gamma) := \sum_{\gamma_i \in \mathcal{E}_n(\gamma)} \omega_n(\gamma_i)\,,$$

where $\mathcal{E}_n(\gamma)$ *denotes the set of all legal paths* γ_i *in* Γ_n *with* $f_n(\gamma_i) = \gamma$.

Proof. We structure this proof into several steps; this structure is not related to the subdivision of the statement of the proposition into the parts (1) and (2) above. Indeed, we prove directly statement (2), as this implies (1) (see subsections 2.5 and 2.7).

(a) We will first show that the definition of $\mu_\Gamma^{\overleftarrow{\varpi}}(\gamma)$ is independent of the level n used in the definition if one supposes n to be high enough (meaning: $\text{minlength}_{\overleftarrow{\Gamma}}(\text{n}) \geqslant |\gamma|$), so that γ traverses at most one intrinsic vertex.

Indeed, for any integer $k \geqslant n$ and any legal path γ'_j in Γ_k with $f_k(\gamma'_j) = \gamma$ there is a legal path γ_i in Γ_n with $f_n(\gamma_i) = \gamma$ such that $\gamma_i = f_{n,k}(\gamma'_j)$.

Thus it suffices to show the following *claim*:

$$\omega_n(\gamma_i) = \sum_{\{\gamma'_j \,|\, f_{n,k}(\gamma'_j) = \gamma_i\}} \omega_k(\gamma'_j) \tag{5.4}$$

(i) If γ_i does not cross over an intrinsic vertex of Γ_n, then any of the γ'_j with $f_{n,k}(\gamma'_j) = \gamma_i$ can not cross either over any intrinsic vertex of Γ_k, as the level maps in any graph tower map intrinsic vertices to intrinsic vertices. In particular, it follows that every short edge e' of Γ_k which is mapped to any short edge e on the path γ_i, must be part of a unique path γ'_j with $f_{n,k}(\gamma'_j) = \gamma_i$.

Furthermore, we obtain $\omega_k(\gamma'_j) = \omega_k(e')$ for any short edge e' contained in γ'_j, and since we similarly have $\omega_n(\gamma_i) = \omega_n(e)$ for any short edge e contained in γ_i, the above claim (5.4) follows now directly from the compatibility condition (5.2).

(ii) In the case that γ_i crosses over a (single !) intrinsic vertex v of Γ_n, then we consider the local edge ε at v used by γ_i, and observe that $\omega(\gamma_i) = \omega(\varepsilon)$ holds, by the definition of $\omega(\gamma_i)$. We note that for any preimage vertex $v' \in \Gamma_k$ of v and any local edge ε' at v' with $f_{n,k}(\varepsilon') = \varepsilon$ there is precisely one edge path γ'_j crossing over ε' such that $f_{n,k}(\gamma'_j) = \gamma_i$. Conversely, for any path γ'_j in Γ_k with $f_{n,k}(\gamma'_j) = \gamma_i$ there must be a preimage vertex v' of v and a local edge ε' at v' with $f_{n,k}(\varepsilon') = \varepsilon$ such that γ'_j crosses over ε'.

In both statements of the preceding paragraph the vertex v' may or may not be intrinsic in the graph Γ_k. However, in any case no other vertex crossed over by γ'_j can be intrinsic, as its image on γ_i would also have to be intrinsic, contradicting the length hypothesis on the edges of Γ_n. Thus, if v' is not intrinsic, then all edges of γ'_j have the same weight as γ'_j, which must hence be equal to $\omega_k(\varepsilon')$, by the switch conditions for ω_k. If v' is intrinsic, then $\omega_k(\gamma'_j) = \omega_k(\varepsilon')$ is given by the definition of the weights of paths. As in the first case (i), the compatibility conditions (5.2) give $\omega_n(\varepsilon) = \sum\limits_{\{\varepsilon' \mid f_{n,k}(\varepsilon')=\varepsilon\}} \omega_k(\varepsilon')$, which shows the above claim (5.4) in this second case (ii).

(b) One needs to verify the Kirchhoff conditions (2.2) for the function $\mu_{\Gamma}^{\overleftarrow{\omega}}$. However, by part (a) of this proof we can consider, for any path γ in Γ of length $|\gamma| = s$, any level graph Γ_n where the minimal length of long edges satisfies $\text{minlength}_{\overleftarrow{\Gamma}}(n) \geqslant s+1$. Then for any legal path γ' in Γ_n with $f_n(\gamma') = \gamma$ the switch conditions (5.1) show directly that $\omega_n(\gamma') = \sum \omega_n(\gamma_i)$, where either the summation is taken over all paths γ_i of length $s+1$ which have γ' as initial subpath, or else over all paths γ_i of length $s+1$ which have γ' as terminal subpath.

The Kirchhoff conditions for the function $\mu_{\Gamma}^{\overleftarrow{\omega}}$ is then a direct consequence of the definition of $\mu_{\Gamma}^{\overleftarrow{\omega}}$.

(c) The condition (2.1) is a direct consequence of the equality $\omega(\overline{e}) = \omega(e)$ from Definition 5.2 (1).

(d) We finally observe that from $\omega_n(\varepsilon_i) = 0$ for any illegal local edge at any level graph Γ_n it follows directly that the support of $\mu^{\overleftarrow{\omega}}$ is contained in $L^{F_N}_{legal}(\overleftarrow{\Gamma})$. □

The converse of Proposition 5.4 is also true, but we don't prove it here, since we will need later a more precise statement (see Proposition 7.4):

Remark 5.5. Let $\overleftarrow{\Gamma} = ((\Gamma_n)_{n\in\mathbb{N}\cup\{0\}}, (f_{m,n})_{0\leqslant m\leqslant n})$ be an expanding graph tower over a marked graph Γ, and let $\mu \in \text{Curr}(F_N)$ be any current over F_N with support satisfying $\text{Supp}(\mu) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma})$.

Then there exists a tower of weight functions $\overleftarrow{\omega} = (\omega_n : \Gamma_n \to \mathbb{R}_{\geqslant 0})_{n\in\mathbb{N}\cup\{0\}}$ on $\overleftarrow{\Gamma}$ which satisfies:

$$\mu^{\overleftarrow{\omega}} = \mu$$

Remark 5.6. It follows directly from Proposition 5.4 that, if one erases from a given marked graph tower $\overleftarrow{\Gamma}$, provided with a weight tower $\overleftarrow{\omega}$, any finite part, then the defined current $\mu^{\overleftarrow{\omega}}$ remains unchanged.

We finish this section with a lemma that will turn out to be rather useful in the section 7.

Lemma 5.7. *Let $\overleftarrow{\Gamma}$ be an expanding graph tower, and let $\overleftarrow{\omega}$ be a weight tower on $\overleftarrow{\Gamma}$.*

Let e be any short edge of Γ_m, and pick any integer $r \geqslant 0$. Let $\mathcal{L}_r(e)$ be the set of all legal paths γ_i in Γ_m which have length $|\gamma_i| = 2r+1$ and have e as central edge.

Consider a second level graph Γ_n of sufficiently high level such that the minimal length of its long edges satisfies $\text{minlength}_{\overleftarrow{\Gamma}}(n) \geqslant 2r+1$. *For any $\gamma_j \in \mathcal{L}_r(e)$ denote by $\mathcal{E}_n(\gamma_j)$ the set of legal paths γ'_i in Γ_n with $f_{m,n}(\gamma'_i) = \gamma_j$. Then we have:*

$$\omega_m(e) = \sum_{\gamma_j\in\mathcal{L}_r(e)} \sum_{\gamma'_i\in\mathcal{E}_n(\gamma'_i)} \omega_n(\gamma'_i)$$

In other words: $\omega_m(e)$ is equal to the sum of all $\omega_n(\gamma'_j)$, where the summation is taken over all legal paths γ'_i in Γ_n with $f_{m,n}(\gamma'_i) \in \mathcal{L}_r(e)$.

Proof. We first observe that for any legal path γ'_i in Γ_n with $f_{m,n}(\gamma'_i) = \gamma_j$ for some $\gamma_j \in \mathcal{L}_r(e)$ the central edge e' of γ'_i satisfies $f_{m,n}(e') = e$.

Conversely, every edge e' in Γ_n with $f_{m,n}(e') = e$ must

be the central edge of some path γ'_i with $f_{m,n}(\gamma'_i) = \gamma_j$ for some of the $\gamma_j \in \mathcal{L}_r(e)$, unless one has $\mathcal{L}_r(e') = \emptyset$. Moreover, any path γ'_i in $\mathcal{L}_r(e')$ must be mapped by $f_{m,n}$ to some $\gamma_j \in \mathcal{L}_r(e)$.

This shows that the set of all edges $e' \in \mathrm{Edges}^{\pm}(\Gamma_n)$ with $f_{m,n}(e') = e$ and $\mathcal{L}_r(e') \neq \emptyset$ coincides precisely with the set of all central edges of legal paths γ'_i in Γ_n which satisfy $f_{m,n}(\gamma'_i) = \gamma_j$ for some of the $\gamma_j \in \mathcal{L}_r(e)$. Thus we obtain the following equality, where $\mathcal{E}_n(e)$ denotes the set of all edges $e' \in \mathrm{Edges}^{\pm}(\Gamma_n)$ with $f_{m,n}(e') = e$, and $\mathcal{E}_n(\gamma_j)$ the set of legal paths γ'_i in Γ_n with $f_{m,n}(\gamma'_i) = \gamma_j$:

$$\sum_{\gamma_j \in \mathcal{L}_r(e)} \sum_{\gamma'_i \in \mathcal{E}_n(\gamma'_i)} \omega_n(\gamma'_i) = \sum_{e' \in \mathcal{E}_n(e)} \sum_{\gamma'_i \in \mathcal{L}_r(e')} \omega_n(\gamma'_i)$$

Now, if e' is sufficiently far away from the intrinsic vertices of Γ_n, then there is only one γ'_i in $\mathcal{L}_r(e')$, and one has $\omega_n(e') = \omega_n(\gamma'_i)$.

Otherwise there are possibly several paths $\gamma'_i \in \mathcal{L}_r(e')$, which by our length assumption $\mathrm{minlength}_{\overleftarrow{\Gamma}}(n) \geqslant 2r + 1$ all pass over the same intrinsic vertex v of Γ_n and over no other intrinsic vertex, so that from the switch conditions on ω_n at this vertex and from our definition of the weights of paths we see directly that $\omega_n(e')$ is equal to the sum of all $\omega_n(\gamma'_i)$ with $\gamma_i \in \mathcal{L}_r(e')$.

Thus we obtain in both cases, for each $e' \in \mathcal{E}_n(e)$, that

$$\omega_n(e') = \sum_{\gamma'_i \in \mathcal{L}_r(e')} \omega_n(\gamma'_i)$$

and hence:

$$\sum_{\gamma_j \in \mathcal{L}_r(e)} \sum_{\gamma'_i \in \mathcal{E}_n(\gamma'_i)} \omega_n(\gamma'_i) = \sum_{e' \in \mathcal{E}_n(e)} \omega_n(e')$$

But from the compatibility conditions (5.2) on weight functions we know that $\omega_m(e)$ is equal to the sum of all $\omega_n(e')$ for any edge e' in Γ_n with $f_{m,n}(e') = e$, which gives the desired conclusion:

$$\sum_{\gamma_j \in \mathcal{L}_r(e)} \sum_{\gamma'_i \in \mathcal{E}_n(\gamma'_i)} \omega_n(\gamma'_i) = \sum_{e' \in \mathcal{E}_n(e)} \omega_n(e') = \omega_m(e)$$

□

Remark 5.8. There is a small delicacy "hidden" in the last proof which we'd like to point out to the reader: In this proof we observed that the compatibility conditions give

$$\omega_m(e) = \sum_{e' \in \mathcal{E}_n(e)} \omega_n(e')\,,$$

and the arguments given there show

$$\omega_n(e') = \sum_{\gamma'_j \in \mathcal{L}_r(e')} \omega_n(\gamma'_j)\,,$$

which gives the desired conclusion

$$\omega_m(e) = \sum_{e' \in \mathcal{E}_n(e)} \sum_{\gamma'_j \in \mathcal{L}_r(e')} \omega_n(\gamma'_j)\,.$$

However, this string of arguments should not induce the reader to believe that the inclusion

$$\bigcup_{e' \in \mathcal{E}_n(e)} \{f_{m,n}(\gamma'_j) \mid \gamma'_j \in \mathcal{L}_r(e')\} \subseteq \mathcal{L}_r(e)$$

is actually an equality. The problem here which needs to be taken into account is that not every legal path in Γ_m lifts necessarily to a legal path in Γ_n, and this applies in particular to the paths in $\mathcal{L}_r(e)$.

6. Graph tower morphisms

Let $\overleftarrow{\Gamma} = ((\Gamma_n)_{n \in \mathbb{N} \cup \{0\}}, (f_{m,n})_{0 \leqslant m \leqslant n})$ and $\overleftarrow{\Gamma}' = ((\Gamma'_n)_{n \in \mathbb{N} \cup \{0\}}, (f'_{m,n})_{0 \leqslant m \leqslant n})$ be two graph towers.

Definition-Remark 6.1. A family of graph maps $\overleftarrow{g} = (g_k : \Gamma_k \to \Gamma'_k)_{k \in \mathbb{N} \cup \{0\}}$ is called a *graph tower morphism* (see Fig. 5), denoted by $\overleftarrow{g} : \overleftarrow{\Gamma} \to \overleftarrow{\Gamma}'$, if

(0) each g_k maps edges to reduced non-trivial edge paths,

(1) the *compatibility equalities*

$$f'_{m,n} g_n = g_m f_{m,n}$$

are satisfied for all integers $n \geqslant m \geqslant 0$, and

(2) each g_k maps paths with infinitely legal f_k-image to paths with infinitely legal f'_k-image (compare Definition 4.4).

In particular, for the infinitely legal symbolic laminations $L^{\Sigma}_{legal}(\overleftarrow{\Gamma})$ and $L^{\Sigma}_{legal}(\overleftarrow{\Gamma}')$ we obtain directly $g'_0(L^{\Sigma}_{legal}(\overleftarrow{\Gamma})) \subseteq L^{\Sigma}_{legal}(\overleftarrow{\Gamma}')$, where g'_0 is the map obtained from g_0 by subdivision of Γ_0 as explained in subsection 2.7 (right before Definition 2.2). If $\overleftarrow{\Gamma}$ and $\overleftarrow{\Gamma}'$ are marked graph towers, and if the map g_0 induces (via the markings) an automorphims $\varphi \in \mathrm{Out}(F_N)$, then we obtain furthermore:

$$\varphi(L^{F_N}_{legal}(\overleftarrow{\Gamma})) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma}')$$

Remark 6.2. One could try to replace condition (2) above by the following:

(2') if each g_k maps legal paths to legal paths.

This condition implies indeed condition (2), but it turns out that it is too strong in practise: On the lowest levels of a graph tower there are in general simply too many legal paths, so that this condition will fail to hold in many interesting cases.

Let $\overleftarrow{\omega} = (\omega_n)_{n \in \mathbb{N} \cup \{0\}}$ be a weight tower on $\overleftarrow{\Gamma}$. We define the image weight tower $\overleftarrow{g}(\overleftarrow{\omega}) = (\omega'_n)_{n \in \mathbb{N} \cup \{0\}}$ on $\overleftarrow{\Gamma}'$ by first subdividing each Γ_n through pulling back via g_n the vertices of Γ'_n to obtain a graph Γ^g_n with edges e^g_i, and with weights $\omega_n(e^g_i) := \omega_n(e)$ if e^g_i results from subdividing the edge e of Γ_n. The map g_n maps every edge e^g_i of Γ^g_n to a single edge of Γ'_n, so that for any edge e' of Γ'_n we can define:

$$\omega'_n(e') = \sum_{\{e^g_i \,\mid\, g_n(e^g_i) = e'\}} \omega_n(e^g_i)$$

$$\begin{tikzcd}
\cdot \arrow[d, dashed] & \cdot \arrow[d, dashed] \\
\cdot \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[d, "f_{n+1,n+2}"'] & \cdot \arrow[d, "f'_{n+1,n+2}"] \\
\Gamma_{n+1} \arrow[r, "g_{n+1}"'] \arrow[d, "f_{n,n+1}"'] & \Gamma'_{n+1} \arrow[d, "f'_{n,n+1}"] \\
\Gamma_n \arrow[r, "g_n"'] \arrow[d] & \Gamma'_n \arrow[d] \\
\cdot \arrow[d, dashed] & \cdot \arrow[d, dashed] \\
\cdot \arrow[d, "f_{0,1}"'] & \cdot \arrow[d, "f'_{0,1}"] \\
\Gamma_0 \arrow[r, "g_0"] & \Gamma'_0
\end{tikzcd}$$

FIGURE 5.

We first verify:

Lemma 6.3. *The weights* ω'_n *define an image weight tower* $\overleftarrow{g}(\overleftarrow{\varpi}) = (\omega'_n)_{n \in \mathbb{N} \cup \{0\}}$ *on* $\overleftarrow{\Gamma}'$*.*

Proof. The compatibility conditions (5.2) and (5.3) are transferred directly from $\overleftarrow{\varpi}$ to $\overleftarrow{g}(\overleftarrow{\varpi})$ by the compatibility equalities in part (1) of Definition-Remark 6.1. For the switch conditions (5.1) we recall from Proposition 5.4 that positive weights are carried only by infinitely legal edge paths. Thus part (2) of Definition-Remark 6.1 allows us to transfer the conditions (5.1) from $\overleftarrow{\varpi}$ to $\overleftarrow{g}(\overleftarrow{\varpi})$. □

On the level of invariant measures, or of currents, a graph tower morphisms gives the following:

Proposition 6.4. *(1) For any graph tower morphism* $\overleftarrow{g} : \overleftarrow{\Gamma} \to \overleftarrow{\Gamma}'$*, with induced homomorphism* $g_{0*} : \pi_1\Gamma_0 \to \pi_1\Gamma'_0$*, and for any weight tower* $\overleftarrow{\varpi}$ *on* $\overleftarrow{\Gamma}$ *one obtains (using the notation introduced in Definition 2.2) for the invariant measures defined by the weight tower* $\overleftarrow{\varpi}$ *and its image* $\overleftarrow{g}(\overleftarrow{\varpi})$*:*

$$(g_{0*}\mu^{\overleftarrow{\varpi}})_{\Gamma'_0} = \mu_{\Gamma'_0}^{\overleftarrow{g}(\overleftarrow{\varpi})}$$

(2) If $\overleftarrow{\Gamma}$ and $\overleftarrow{\Gamma}'$ are marked, and if g_{0*} induces via the markings an isomorphism $\varphi \in Out(F_N)$, then one obtains (compare Proposition 2.1):

$$\varphi(\mu^{\overleftarrow{\omega}}) = \mu^{\overleftarrow{g}(\overleftarrow{\omega})}$$

Proof. The proof is a direct consequence of the above definition of the image weight tower $g(\overleftarrow{\omega})$, together with the definition of the Kolmogorov function $\mu^{\overleftarrow{\omega}}$ in Proposition 5.4 and the transition of Kolmogorov functions under the given legality assumptions spelled out in Definition 2.2 and Proposition 2.1. ◻

7. Uniqueness conditions

In this section we will use the same language as in section 5. In particular we will assume throughout this section Convention 5.1.

Definition 7.1. Let $\overleftarrow{\Gamma} = ((\Gamma_n)_{n \in \mathbb{N} \cup \{0\}}, (f_{m,n})_{0 \leqslant m \leqslant n})$ be a graph tower. A level graph Γ_n is called *non-repeating* if there exists an integer

$$\mathrm{repbound}_{\overleftarrow{\Gamma}}(n) \geqslant 0\,,$$

called *repetition bound* (and sometimes "abbreviated" to $r(n)$), such that any two legal edge paths γ and γ' which "read off" the same path $f_n(\gamma) = f_n(\gamma')$ of length $|f_n(\gamma)| = |f_n(\gamma')| = 2\,\mathrm{repbound}_{\overleftarrow{\Gamma}}(n) + 1$ must coincide in their middle edge.

The tower $\overleftarrow{\Gamma}$ is called *non-repeating* if every level graph Γ_n is non-repeating.

Remark 7.2. (1) In [16] verifiable combinatorial conditions have been exhibited which ensure that $\overleftarrow{\Gamma}$ is non-repeating.

(2) The following special case, however, is easy to deduce from the definitions: If for some level graph Γ_n of a graph tower $\overleftarrow{\Gamma}$ the map $f_n : \Gamma_n \to \Gamma_0$ induces an isomorphism on π_1, then the level graph Γ_n is non-repeating.

Lemma 7.3. *Let $\overleftarrow{\Gamma}$ be a graph tower as before, and let $\overleftarrow{\omega}$ be a weight tower on $\overleftarrow{\Gamma}$.*

For some integers $n \geqslant m \geqslant 0$ assume that the level graph Γ_m is non-repeating with repetition bound $r(m) := \mathrm{repbound}_{\overleftarrow{\Gamma}}(m)$, and assume that

$$\mathrm{minlength}_{\overleftarrow{\Gamma}}(n) \;\geqslant\; 2\,\mathrm{repbound}_{\overleftarrow{\Gamma}}(m) + 1$$

(i.e. for any long edge e_i^* of Γ_n^* one has $|f_n(e_i^*)| \geqslant 2\,\mathrm{repbound}_{\overleftarrow{\Gamma}}(m) + 1$).

Let e be any short edge of Γ_m, and consider the set $\mathcal{L}_{r(m)}(e)$ of all legal paths γ_i of length $|\gamma_i| = 2r(m)+1$ with e as central edge. Then $\omega_m(e)$ is equal to the sum of all $\omega_n(\gamma'_i)$, where the summation is taken over all legal paths γ'_i in Γ_n with $f_n(\gamma'_i) = f_m(\gamma_j)$ for any $\gamma_j \in \mathcal{L}_{r(m)}(e)$.

Proof. The claim follows directly from Lemma 5.7, for the specification $r = \mathrm{repbound}_{\overleftarrow{\Gamma}}(m)$, if one can show that the following two sets are equal:

1. the set of all legal paths γ'_i in Γ_n with $f_n(\gamma'_i) = f_m(\gamma_j)$ for any $\gamma_j \in \mathcal{L}_{r(m)}(e)$, and
2. the set of all legal paths γ'_i in Γ_n with $f_{m,n}(\gamma'_i) \in \mathcal{L}_{r(m)}(e)$.

However, the equality of these two sets is a direct consequence of the definition of the repetition bound $\mathrm{repbound}_{\overleftarrow{\Gamma}}(m)$. ◻

Proposition 7.4. *(1) Let $\overleftarrow{\Gamma}$ be an non-repeating expanding graph tower, and let μ_Σ an invariant measure on the infinitely legal symbolic lamination $L^{\Sigma}_{legal}(\overleftarrow{\Gamma})$. Then there is a unique weight tower $\overleftarrow{\omega} = \overleftarrow{\omega}^{\mu_\Sigma}$ on $\overleftarrow{\Gamma}$ for which the associated invariant measure $\mu_\Sigma^{\vec{\omega}}$ satisfies:*

$$\mu_\Sigma^{\vec{\omega}} = \mu_\Sigma$$

If $\overleftarrow{\Gamma}$ is marked, then we can assume that μ_Σ is given by any current $\mu \in Curr(F_N)$ with support $Supp(\mu) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma})$; in this case there is a unique weight tower $\overleftarrow{\omega} = \overleftarrow{\omega}^{\mu}$ on $\overleftarrow{\Gamma}$ for which the associated current $\mu^{\vec{\omega}}$ satisfies:

$$\mu^{\vec{\omega}} = \mu$$

(2) More precisely, using the Kolmogorov function μ_Γ (for $\Gamma := \Gamma_0$) given by μ (or by μ_Σ), the unique weight tower $\overleftarrow{\omega} = (\omega_n^\mu)_{n \in \mathbb{N} \cup \{0\}}$ is given by the following formula:

$$\omega_m^\mu(e) = \sum_{\{f_m(\gamma_j) \,|\, \gamma_j \in \mathcal{L}_{r(m)}(e)\}} \mu_\Gamma(f_m(\gamma_j))$$

Proof. Since this proof is a bit lengthly, and also rather delicate in some of its arguments, we are asking the reader to be careful, in each of the following 4 parts below, where we prove successively

(1) the uniqueness of ω (assuming $\mu^{\vec{\omega}} = \mu$),
(2) that ω_m^μ satisfies the switch conditions,
(3) that the ω_m^μ are compatible, and
(4) that the current defined by $\overleftarrow{\omega}^\mu$ is equal to μ.

(1) From the definition of $\mu^{\vec{\omega}}$ in section 5 we know that for any path γ in Γ the value of the Kolmogorov function $\mu_\Gamma^{\vec{\omega}}(\gamma)$ is given as sum of all $\omega_n(\gamma_i)$, where γ_i is any legal path in Γ_n with $f_n(\gamma_i) = \gamma$, assuming that n is sufficiently large to guarantee $|\gamma| \leqslant \text{minlength}_{\overleftarrow{\Gamma}}(n)$ (i.e. $|\gamma| \leqslant |e^*|$ for any long edge e^* of Γ_n).

We now consider any edge e of any level graph Γ_m, and observe that for any n with $\text{minlength}_{\overleftarrow{\Gamma}}(n) \geqslant 2r(m)+1$, according to the previous paragraph, for any path $\gamma_j \in \mathcal{L}_{r(m)}(e)$ the value $\mu_\Gamma^{\vec{\omega}}(f_m(\gamma_j))$ is given as sum of all $\omega_n(\gamma_i')$, where γ_i' is any legal path in Γ_n with $f_n(\gamma_i') = f_m(\gamma_j)$.

Hence we obtain directly from Lemma 7.3 that the weight $\omega_m(e)$ is equal to the sum of all $\mu_\Gamma^{\vec{\omega}}(f_m(\gamma_j))$ for all $\gamma_j \in \mathcal{L}_{r(m)}(e)$. This shows that any weight tower $\overleftarrow{\omega}$ on $\overleftarrow{\Gamma}$ which satisfies $\mu^{\overleftarrow{\omega}} = \mu$ must satisfy

$$\omega_m(e) = \sum_{\{f_m(\gamma_j \,|\, \gamma_j \in \mathcal{L}_{r(m)}(e)\}} \mu_\Gamma(f_m(\gamma_j))$$

and hence is indeed determined by the current μ (or rather, by its Kolmogorov function μ_Γ associated to Γ).

To alert the reader, we would like to be specific in that in general the following two sums have different values:

$$\sum_{\{f_m(\gamma_j \,|\, \gamma_j \in \mathcal{L}_{r(m)}(e)\}} \mu_\Gamma(f_m(\gamma_j)) \neq \sum_{\gamma_j \in \mathcal{L}_{r(m)}(e)} \mu_\Gamma(f_m(\gamma_j))$$

The reason why we have to work with the first and not with the second sum is that distinct $\gamma_j, \gamma_{j'} \in \mathcal{L}_{r(m)}(e)$ may well map to equal paths $f_m(\gamma_j) = f_n(\gamma_{j'})$, so that in the second sum there is a potential double counting that one has to avoid.

Aside: The above use of Lemma 7.3 relies on the hypothesis that Γ_m be non-repeating. Indeed, without this hypothesis the last equality would in general be wrong, for weight towers $\overleftarrow{\varpi}$ with $\mu^{\overleftarrow{\varpi}} = \mu$, as would indeed be the statement of our proposition to be proved.

We will show next (parts (2) and (3) below) that if $\mathrm{Supp}(\mu) \subseteq L^{F_N}_{legal}(\vec{\Gamma})$, then using the last equality as definition for $\omega^\mu_m(e)$ for any edge e of any level graph Γ_m, one obtains indeed a weight tower.

(2) For our first purpose, to show that the functions ω^μ_m satisfy the switch conditions, we extend the given definition of ω^μ_m to the local edges ε_i at any vertex v of Γ_m:

$$\omega^\mu_m(\varepsilon_i) := \sum_{\{f_m(\gamma_j) \,|\, \gamma_j \in \mathcal{L}_{r(m+1)}(\varepsilon_i)\}} \mu_\Gamma(f_m(\gamma_j))$$

Here $\mathcal{L}_{r(m+1)}(\varepsilon_i)$ denotes the set of all legal paths of length $2r(m)+2$ with center vertex v.

We now consider the case where v is the endpoint of a given edge e, and observe that for any path $\gamma \in \mathcal{L}_{r(m)}(e)$ there is precisely one local edge ε_i at v with initial local vertex $v(\overline{e})$, namely the local edge at v which is crossed by γ, such that any legal prolongation of γ of length $|\gamma|+1$ to an edge path with γ as initial subpath gives a path γ_j in $\mathcal{L}_{r(m+1)}(\varepsilon_i)$.

Conversely, for any local edge ε_i with initial vertex $v(\overline{e})$ any path γ_j in $\mathcal{L}_{r(m+1)}(\varepsilon_i)$ must have some $\gamma \in \mathcal{L}_{r(m)}(e)$ as initial subpath, with $|\gamma_j| = |\gamma| + 1$.

Furthermore, any legal path γ_1 in Γ_m which contains any subpath γ' with $f_m(\gamma') = f_m(\gamma)$ for any $\gamma \in \mathcal{L}_{r(m)}(e)$ must pass through the edge e, by the non-repetitiveness hypothesis on Γ_m, so that in fact we have $\gamma' \in \mathcal{L}_{r(m)}(e)$.

As a consequence, we see that the union of all $\mathcal{L}_{r(m+1)}(\varepsilon_i)$, where ε_i is any local edge with initial vertex $\iota(\varepsilon)$ equal to the local vertex $v(\overline{e})$, coincides precisely with the set of all legal paths γ_j in Γ_m of length $2r(m)+2$ which contain as initial subpath any path γ'_j with $f_m(\gamma'_j) = f_m(\gamma)$ for any $\gamma \in \mathcal{L}_{r(m)}(e)$.

Thus we can apply the Kirchhoff rules of the Kolmogorov function μ_Γ to the set of paths $f(\gamma)$ in Γ, for any $\gamma \in \mathcal{L}_{r(m)}(e)$, to obtain:

$$\sum_{\{f_m(\gamma) \,|\, \gamma \in \mathcal{L}_{r(m)}(e)\}} \mu_\Gamma(f_m(\gamma)) = \sum_{\{\varepsilon_i \,|\, \iota(\varepsilon_i) = v(\overline{e})\}} \sum_{\{f_m(\gamma_j) \,|\, \gamma_j \in \mathcal{L}_{r(m+1)}(\varepsilon_i)\}} \mu_\Gamma(f_m(\gamma_j))$$

But by our definition of $\omega^\mu_m(e)$ and $\omega^\mu_m(\varepsilon_i)$ this shows precisely that the function ω^μ_m satisfies the switch condition at the local vertex $v(\overline{e})$.

Notice that to ensure the switch conditions for the weight functions ω^μ_m we have not used that the support of μ is contained in the infinitely legal lamination $L^{F_N}_{legal}(\overleftarrow{\Gamma})$. However, to ensure compatibility of the weight functions, this is necessary, and also sufficient:

(3) One first notes, for any $r \geqslant \mathrm{repbound}_{\overleftarrow{\Gamma}}(m)$ and any path β in Γ of length $2r+1$, the following: if there is a path $\gamma_j \in \mathcal{L}_{r(m)}(e)$ such that in β the central segment β' of length $2\,\mathrm{repbound}_{\overleftarrow{\Gamma}}(m)+1$ satisfies $\beta' = f_m(\gamma_j)$, then either there is a legal path γ_k in Γ_m with $f_m(\gamma_k) = \beta$, or else $\mu_\Gamma(\beta) = 0$, by the hypothesis $\mathrm{Supp}(\mu) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma})$. Moreover, from non-repetitiveness hypothesis on Γ_m we obtain that γ_k and γ_j must coincide in their central edge, i.e. $\gamma_k \in \mathcal{L}_r(e)$.

Hence we obtain:

$$\sum_{\{f_m(\gamma_j) \,|\, \gamma_j \in \mathcal{L}_{r(m)}(e)\}} \mu_\Gamma(f_m(\gamma_j)) = \sum_{\{f_m(\gamma_k) \,|\, \gamma_k \in \mathcal{L}_r(e)\}} \mu_\Gamma(f_m(\gamma_k))$$

One now specifies, for any $n \geqslant m$ the above value $r = \text{ repbound}_{\overleftarrow{\Gamma}}(n)$ (noting $\text{repbound}_{\overleftarrow{\Gamma}}(n) \geqslant \text{repbound}_{\overleftarrow{\Gamma}}(m)$ as direct consequence of the definition of the repetition bound) and observes that for each $\gamma_k \in \mathcal{L}_{r(n)}(e)$ in Γ_m all legal paths γ'_h in Γ_n with $f_{m,n}(\gamma'_h) = \gamma_k$ must coincide in their central edge e', which then satisfies $f_{m,n}(e') = e$. Furthermore, again by the hypothesis $\text{Supp}(\mu) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma})$, at least one such γ'_h must exist, or else one has $\mu_\Gamma(f_m(\gamma_k)) = 0$. Last, by the non-repetitiveness of Γ_n, for any distinct edges $e' \neq e'' \in \mathcal{E}_n(e)$ and any $\gamma'_k \in \mathcal{L}_{r(n)}(e')$ and $\gamma''_k \in \mathcal{L}_{r(n)}(e''')$ one has $f_n(\gamma'_k) \neq f_n(\gamma''_k)$.

Hence we obtain

$$\sum_{e' \in \mathcal{E}_n(e)} \sum_{\{f_n(\gamma_h) \,|\, \gamma'_h \in \mathcal{L}_{r(n)}(e')\}} \mu_\Gamma(f_n(\gamma_h)) =$$

$$\sum_{\{f_m(\gamma_k) \,|\, \gamma_k \in \mathcal{L}_{r(n)}(e)\}} \mu_\Gamma(f_m(\gamma_k)) = \sum_{\{f_m(\gamma_j) \,|\, \gamma_j \in \mathcal{L}_{r(m)}(e)\}} \mu_\Gamma(f_m(\gamma_j))$$

and thus

$$\sum_{e' \in \mathcal{E}_n(e)} \omega^\mu_n(e') = \omega^\mu_m(e)$$

.

(4) It remains to show that the current $\mu' := \mu^{\overleftarrow{\omega}^\mu}$ defined by the weight tower $\overleftarrow{\omega}^\mu$ is identical with the originally given current μ. Because of the Kirchhoff conditions, it suffices to show $\mu'_\Gamma(\beta) = \mu_\Gamma(\beta)$ for paths β in Γ of odd length. For any such path β one obtains $\mu'(\beta)$ by considering, a level graph Γ_n with $\text{minlength}_{\overleftarrow{\Gamma}}(n) \geqslant |\beta|$, and the set $\mathcal{E}_n(\beta)$ of all legal paths γ_k in Γ_n with $f_n(\gamma_k) = \beta$.

By definition of μ' we have $\mu'_\Gamma(\beta) = \sum\limits_{\gamma_k \in \mathcal{E}_n(\beta)} \omega^\mu_n(\gamma_k)$, with $\omega^\mu_n(\gamma_k) = \omega^\mu_n(e_k)$ for any arbitrarily chosen (short) edge e_k contained in γ_k, in case that γ_k doesn't cross over any intrinsic vertex of Γ_n. In the case where ε_k is the local edge traversed by γ_k at the only intrinsic vertex v crossed by γ_k, then we have $\omega^\mu_n(\gamma_k) = \omega^\mu_n(\varepsilon_k)$. But by definition of ω^μ_n we obtain $\omega^\mu_n(e_k) = \sum\limits_{\{f_n(\gamma_j) \,|\, \gamma_j \in \mathcal{L}_{r(n)}(e_k)\}} \mu_\Gamma(f_n(\gamma_j))$ and $\omega^\mu_n(\varepsilon_k) = \sum\limits_{\{f_n(\gamma_j) \,|\, \gamma_j \in \mathcal{L}_{r(n+1)}(\varepsilon_k)\}} \mu_\Gamma(f_n(\gamma_j))$. Hence we obtain, for $\mathcal{L}(\gamma_k) := \mathcal{L}_{r(n)}(e_k)$ or $\mathcal{L}(\gamma_k) := \mathcal{L}_{r(n)+1}(\varepsilon_k)$, the following equalities:

$$\mu'_\Gamma(\beta) = \sum_{\gamma_k \in \mathcal{E}_n(\beta)} \omega^\mu_n(\gamma_k) = \sum_{\gamma_k \in \mathcal{E}_n(\beta)} \sum_{\{f_n(\gamma_j) \,|\, \gamma_j \in \mathcal{L}(\gamma_k)\}} \mu(f_n(\gamma_j))$$

In both cases, we observe that γ_k is contained as subpath in any of the $\gamma_j \in \mathcal{L}_{r(n)}(e_k)$ or of the $\gamma_j \in \mathcal{L}_{r(n+1)}(\varepsilon_k)$, modulo possibly replacing $r(n)$ or $r(n)+1$ by a suitable larger bound $r \geqslant r(n)$, which does not change the value of the sum $\sum \mu_\Gamma(f_n(\gamma_j))$, as we have shown above (for short edges e, but the same proof also applies to local edges ε).

On the other hand, by definition of the set $\mathcal{E}_n(\beta)$, any sufficiently long legal path in Γ_n which is mapped by f_n to a path which contains β, must itself contain some of the $\gamma_k \in \mathcal{E}_n(\beta)$ as subpath at the corresponding locus, and hence also one of the prolongations $\gamma_j \in \mathcal{L}'(\gamma_k)$ of γ_k, with $\mathcal{L}'(\gamma_k) = \mathcal{L}_r(e_k)$ or $\mathcal{L}'(\gamma_k) = \mathcal{L}_r(\varepsilon_k)$ given as before. This shows, by the Kirchhoff conditions for μ_Γ and the fact that the support of μ is contained in the set of legal paths in Γ_n, that we have

$$\mu(\beta) = \sum_{\gamma_k \in \mathcal{L}_n(\beta)} \sum_{\{f_n(\gamma_j) \,|\, \gamma_j \in \mathcal{L}'(\gamma_k)\}} \mu(f_n(\gamma_j))$$

and thus $\mu'_\Gamma(\beta) = \mu(\beta)$. □

8. Weight vectors

Convention 8.1. In this section we assume again Convention 5.1, i.e. any graph tower $\overleftarrow{\Gamma} = ((\Gamma_n)_{n\in\mathbb{N}\cup\{0\}}, (f_{m,n})_{0\leqslant m\leqslant n})$ is given in short-edge dialect. However, for most of this section will work with the associated long-edge graph tower $((\Gamma^*_n)_{n\in\mathbb{N}\cup\{0\}}, (f^*_{m,n})_{0\leqslant m\leqslant n})$ from Remark 4.2.

For any of the graph maps $f^*_{m,n}$ of a graph tower $\overleftarrow{\Gamma}$ we can define a non-negative *transition matrix*

$$M(f^*_{m,n}) = (m_{e^*,e'^*})_{e^*\in\mathrm{Edges}^+(\Gamma^*_m),\, e'^*\in\mathrm{Edges}^+(\Gamma^*_n)}$$

of $f^*_{m,n}$, which is defined by setting m_{e^*,e'^*} equal to the number of times that $f^*_{m,n}(e'^*)$ crosses over e^* or over $\overline{e}^*$ (in both cases counted positively), see subsection 2.2. From the compatibility condition for graph towers (Definition 4.1 (c)) one derives directly that

$$M(f^*_{k,n}) = M(f^*_{k,m})M(f^*_{m,n})$$

holds for all integers $n \geqslant m \geqslant k \geqslant 0$.

For any weight function ω^*_n on a long-edge level graph Γ^*_n, induced as described in Remark 5.3 (2) by a weight function ω_n on the short-edge level graph Γ_n, we consider the associated *weight vector* $\vec{v}^{\omega_n} := (\omega^*_n(e^*_i))_{e^*_i\in\mathrm{Edges}^+(\Gamma^*_n)}$, thought of as column vector. We deduce from the compatibility conditions (5.2) that for any weight tower $\overleftarrow{\omega} = (\omega_n)_{n\in\mathbb{N}\cup\{0\}}$ on $\overleftarrow{\Gamma}$, and for any integers $n \geqslant m \geqslant 0$, the associated weight vectors satisfy the following equations:

$$\vec{v}^{\omega_m} = M(f^*_{m,n})\vec{v}^{\omega_n}$$

This gives rise to the following:

Definition 8.2. A *vector tower* $\overleftarrow{v}$ on a given graph tower $\overleftarrow{\Gamma}$ is a family $\overleftarrow{v} = (\vec{v}_n)_{n\in\mathbb{N}\cup\{0\}}$ of functions $\vec{v}_n : \mathrm{Edges}^+(\Gamma^*_n) \to \mathbb{R}_{\geqslant 0}$ on the set of oriented long edges of the level graphs Γ^*_n of $\overleftarrow{\Gamma}$. The functions $\vec{v}_n$ are thought of as column vectors $\vec{v}_n = (\vec{v}(e^*_i))_{e^*_i\in\mathrm{Edges}^+(\Gamma^*_n)}$, and they must satisfy the *compatibility equalities*

$$\vec{v}_m = M(f^*_{m,n})\vec{v}_n$$

for all $n \geqslant m \geqslant 0$.

Remark 8.3. If $\overleftarrow{\Gamma}$ is an expanding graph tower, and if $\overleftarrow{v} = (\vec{v}_n)_{n\in\mathbb{N}\cup\{0\}}$ is a vector tower on $\overleftarrow{\Gamma}$, then we have:

(1) For any constant $C > 0$, any level $m \geqslant 0$, and and any sufficiently large difference $n - m$, the matrix $M(f^*_{m,n})$ has in every column (i.e. for every edge e'^* of Γ^*_n) a coefficient $m_{e^*,e'^*} > C$.
(2) As a consequence, we observe (using the equality from Definition 8.2):

$$\lim_{n\to\infty} \max\{v_n(e^*) \mid e^* \in \mathrm{Edges}^+(\Gamma^*_n)\} = 0$$

Proposition 8.4. *Let $\overleftarrow{\Gamma} = ((\Gamma_n)_{n\in\mathbb{N}\cup\{0\}}, (f_{m,n})_{0\leqslant m\leqslant n})$ be an expanding graph tower, and assume furthermore that the number of intrinsic vertices of any level graph Γ_n is bounded above independently of n.*

Then any vector tower $\overleftarrow{v} = (\vec{v}_n)_{n\in\mathbb{N}\cup\{0\}}$ on $\overleftarrow{\Gamma}$ determines a weight tower $\overleftarrow{\omega}(\overleftarrow{v}) = (\omega^*_n)_{n\in\mathbb{N}\cup\{0\}}$ on $\overleftarrow{\Gamma}$, given by

$$\omega^*_n(e^*) = \omega^*_n(\overline{e}^*) := \vec{v}_n(e^*)$$

for all long edges $e^* \in \mathrm{Edges}^+(\Gamma^*_n)$ and any $n \geqslant 0$. In particular, the weight tower $\overleftarrow{v}$ determines an invariant measure

$$\mu_\Sigma^{\overleftarrow{v}} := \mu_\Sigma^{\overleftarrow{\omega}(\overleftarrow{v})}$$

on $L^\Sigma_{legal}(\overleftarrow{\Gamma})$, and also, if $\overleftarrow{\Gamma}$ is provided with a marking, a current

$$\mu^{\overleftarrow{v}} := \mu^{\overleftarrow{\omega}(\overleftarrow{v})} \in Curr(F_N)$$

which has support in $L^{F_N}_{legal}(\overleftarrow{\Gamma})$.

Proof. Every vector $\vec{v}_n$ determines via $\omega^*_n(e^*) = \omega^*_n(\overline{e}^*) := \vec{v}_n(e^*)$ a non-negative function ω^*_n on the edges of Γ^*_n. We extend this function to the local edges ε_i of Γ^*_n by defining

$$\omega_n(\varepsilon_i) := \sup_{t \geqslant n} \sum_{e'^*_k \in \mathrm{Edges}^+(\Gamma^*_t)} m^t_{i,k}\, \omega^*(e'^*_k)\,,$$

where $m^t_{i,k}$ denotes the number of times that $f^*_{n,t}(e'^*_k)$ crosses over ε_i or $\overline{\varepsilon}_i$. From the compatibility equalities on the $\vec{v}_n$ we obtain the compatibility conditions for the functions ω^*_n (or rather, more precisely, for the short-edge weight function ω_n): The equality (5.2) from Definition 5.2 (2) follows directly from our assumption $\vec{v}_m = M(f^*_{m,n})\vec{v}_n$ in Definition 8.2, while equality (5.3) is a direct consequence of the above definition of the $\omega_n(\varepsilon_i)$. It is easy to see that this definition also implies directly the switch conditions (5.1) for ω_n, up to a possible error that comes from the fact that an occurrence of ε_i in that path $f^*_{n,t}(e'^*_k)$ may be initial or terminal.

However, from the property (2) of Remark 8.3 we deduce that any possible discrepancy in the switch conditions at any vertex of the level graph Γ^*_n must tend to 0, for $n \to \infty$.

On the other hand, the total sum over all switch condition discrepancies at the vertices of Γ^*_n must be a non-strictly decreasing function of n, as follows directly from the compatibility conditions $\vec{v}_m = M(f^*_{m,n})\vec{v}_n$. Hence the hypothesis of a uniform bound on the number of vertices at any level implies directly that the above error term in the switch conditions must be zero, so that the family of all ω^*_n defines indeed a weight tower $\overleftarrow{\omega}(\overleftarrow{v})$ on $\overleftarrow{\Gamma}$. Hence, by Proposition 5.4, $\overleftarrow{v}$ determines an invariant measure μ_Σ on $L^\Sigma_{legal}(\overleftarrow{\Gamma})$, and also, in case that $\overleftarrow{\Gamma}$ is marked, a current $\mu^{\overleftarrow{v}} := \mu^{\overleftarrow{\omega}(\overleftarrow{v})} \in \mathrm{Curr}(F_N)$ which has support in $L^{F_N}_{legal}(\overleftarrow{\Gamma})$. □

Remark 8.5. The reader observes easily that the compatibility condition $\vec{v}_m = M(f^*_{m,n})\vec{v}_n$ is equivalent to equality (5.2). Hence the above proof shows that for expanding graph towers, with uniform bound on the number of intrinsic vertices of the level graphs, the equality (5.3) as well as equality (5.1) is actually a consequence of equality (5.2), a fact which is at least at first sight not immediately obvious.

Remark 8.6. From fact (2) of Remark 8.3, that $\omega_n(e)$ for any single edge e of the level graph Γ_n tends to 0 with increasing n, one can also deduce that every current μ with support in $L^{F_N}_{legal}(\overleftarrow{\Gamma})$ actually has its support in the sublamination $L^{F_N}_{used}(\overleftarrow{\Gamma}) \subseteq L^{F_N}_{legal}(\overleftarrow{\Gamma})$ (see Remark 4.5). Similarly, any invariant measure μ_Σ on $L^\Sigma_{legal}(\overleftarrow{\Gamma})$ has in fact its support on $L^\Sigma_{used}(\overleftarrow{\Gamma})$.

We finish this section by considering the behavior of vector towers under a graph tower morphism $\overleftarrow{g} : \overleftarrow{\Gamma} \to \overleftarrow{\Gamma}'$ as defined in section 6.

We first notice that every level graph map $g_k : \Gamma_k \to \Gamma'_k$ defines a *level transition matrix* $M(g^*_k) = (m_{e'^*, e^*})_{e'^* \in \mathrm{Edges}^+(\Gamma'^*_k),\, e^* \in \mathrm{Edges}^+(\Gamma^*_k)}$, where $m_{e'^*, e^*}$ is the number of times that $g^*_k(e^*)$

crosses over e'^* or over $\overline{e}'^*$. This definition, together with the compatibility equalities in Definition 6.1, yields directly

$$M(g_k^*)M(f_{k,k+1}^*) = M(f'^*_{k,k+1})M(g_{k+1}^*)$$

and thus:

Proposition 8.7. *For any graph tower morphism $\overleftarrow{g} : \overleftarrow{\Gamma} \to \overleftarrow{\Gamma}'$ and any vector tower $\overleftarrow{v} = (\vec{v}_n)_{n\in\mathbb{N}\cup\{0\}}$ on $\overleftarrow{\Gamma}$, the family of vectors $\vec{v}\,'_n := M(g_n^*)\vec{v}_n$ defines a vector tower $\overleftarrow{v}'$ on $\overleftarrow{\Gamma}'$.*

Proof. According to Definition 8.2 it suffices to verify, for all integers $n \geqslant m \geqslant 0$:

$$M(f'^*_{m,n})\vec{v}\,'_n = M(f'^*_{m,n})M(g_n^*)\vec{v}_n = M(g_m^*)M(f'^*_{m,n})\vec{v}_n = M(g_m^*)\vec{v}_m = \vec{v}\,'_m$$

□

We call the vector tower $\overleftarrow{v}' = (\vec{v}\,'_n)_{n\in\mathbb{N}\cup\{0\}}$ the *image vector tower* of $\overleftarrow{v}$ under the morphisms $\overleftarrow{g}$ and denote it by $\overleftarrow{g}(\overleftarrow{v})$.

As a direct consequence of the last proposition and of Proposition 6.4 we obtain for the currents $\mu^{\overleftarrow{v}}$ and $\mu^{\overleftarrow{g}(\overleftarrow{v})}$ defined (via Proposition 8.4) by $\overleftarrow{v}$ and its image $\overleftarrow{g}(\overleftarrow{v})$ respectively:

Proposition 8.8. *(1) For any graph tower morphism $\overleftarrow{g} : \overleftarrow{\Gamma} \to \overleftarrow{\Gamma}'$ and for any vector tower $\overleftarrow{v} = (\vec{v}_n)_{n\in\mathbb{N}\cup\{0\}}$ on $\overleftarrow{\Gamma}$ one has:*

$$\overleftarrow{\omega}(\overleftarrow{g}(\overleftarrow{v})) = \overleftarrow{g}(\overleftarrow{\omega}(\overleftarrow{v}))$$

(2) The homomorphism $g_{0,*} : \pi_1\Gamma_0 \to \pi_1\Gamma'_0$ maps the invariant measure $\mu_\Sigma^{\overleftarrow{v}} = \mu_\Sigma^{\overleftarrow{\omega}(\overleftarrow{v})}$ on $L^\Sigma_{legal}(\overleftarrow{\Gamma})$ to an invariant measure $g_{0,*}\mu_\Sigma^{\overleftarrow{v}}$ on $L^\Sigma_{legal}(\overleftarrow{\Gamma}')$ which satisfies:

$$g_{0,*}\mu_\Sigma^{\overleftarrow{v}} = \mu_\Sigma^{\overleftarrow{g}(\overleftarrow{v})}$$

(3) If $\overleftarrow{\Gamma}$ and $\overleftarrow{\Gamma}'$ are marked graph towers, and if the map $g_0 : \Gamma_0 \to \Gamma'_0$ induces (via the marking isomorphisms) an automorphisms $\varphi \in Out(F_N)$, one has furthermore:

$$\varphi(\mu^{\overleftarrow{v}}) = \mu^{\overleftarrow{g}(\overleftarrow{v})}$$

□

9. Train track maps

We now consider any expanding train track map $f : \Gamma \to \Gamma$ which represents an isomorphism $\varphi \in \mathrm{Out}(F_N)$ via some marking isomorphisms $\theta : \pi_1\Gamma \xrightarrow{\cong} F_N$ (see subsection 2.2 for the terminology). Recall also that we use in this paper the term "f-legal" for what is in the train track literature usually called "legal".

Since we want to work below, as in the previous section, in long-edge dialect, we suppress all valence 2 vertices from Γ, i.e. we assume $\Gamma = \Gamma^*$.

In order to derive a graph tower from $f : \Gamma \to \Gamma$ we define level graphs $\Gamma_n := \Gamma$ and graph maps $f_{m,n} := f^{n-m}$, for all integers $n \geqslant m \geqslant 0$. From our definitions in section 4 and from Remark 7.2 (2) we obtain directly:

Proposition 9.1. *(1) For any expanding train track map $f : \Gamma \to \Gamma$ the families of level graphs $\Gamma_n := \Gamma$ and graph maps $f_{m,n} := f^{n-m}$ defines a marked graph tower $\overleftarrow{\Gamma}_f$.*
(2) Furthermore one has:

(a) $\overleftarrow{\Gamma}_f$ *is expanding:* $\text{minlength}_{\overleftarrow{\Gamma}_f}(n) \to \infty$ for $n \to \infty$
(b) *The number of intrinsic vertices of* Γ_n *is independent of* n.
(c) *If* f *is a homotopy equivalence, then every* Γ_n *is non-repeating.*

□

Remark 9.2. (1) We observe that a (possibly infinite or biinfinite) path γ in Γ_0 is infinitely legal with respect to the graph tower $\overleftarrow{\Gamma}_f$ (i.e. $\gamma \in \mathcal{P}_{legal}(\overleftarrow{\Gamma})$, see Definition 4.4) if and only if the corresponding path in Γ is *infinitely* f*-legal*, i.e. it is the f^t-image of some f-legal path, for any integer $t \geqslant 0$ (see subsection 2.2). This shows $L^{F_N}_\infty(f) = L^{F_N}_{legal}(\overleftarrow{\Gamma}_f)$.

(2) Recall from subsection 2.2 that for any expanding train track map $f : \Gamma \to \Gamma$ any infinitely f-legal path is mapped by f to an infinitely f-legal path.

Remark 9.3. The reader who is familiar with the train track technology for automorphisms of free groups will observe that, in the special case where f represents an iwip automorphism of F_N, one deduces from the above construction that the lamination $L^{F_N}_{used}(\overleftarrow{\Gamma}_f)$ from Remark 4.5 coincides precisely with the Bestvina-Handel-Feighn's "attracting" lamination $L^{F_N}_{BFH}(f)$ generated by the paths $f^n(e)$ for any edge e of Γ (compare [15]).

We define a graph tower morphism $\overleftarrow{g}$ (as defined through properties (1) and (2) of Definition-Remark 6.1) from $\overleftarrow{\Gamma}_f$ to itself, by setting the level maps $g_n : \Gamma_n \to \Gamma_n$ equal to f, via the above identification $\Gamma_n = \Gamma$. Indeed, property (1) follows immediately from the compatibility equalities for $\overleftarrow{\Gamma}_f$ (as all maps concerned are powers of f). For property (2) we observe that paths with infinitely legal f_n-image are mapped by g_n to paths with infinitely legal f_n-image, by parts (1) and (2) of the above Remark 9.2.

Let now $\vec{v}$ be a non-negative eigenvector with eigenvalue $\lambda > 1$ of the transition matrix $M(f)$ for the train track map f of Γ (see section 2). Via the identification $\Gamma_n = \Gamma$ for all level graphs of $\overleftarrow{\Gamma}_f$ we define level vectors $\vec{v}_n = \frac{1}{\lambda^n}\vec{v}$. From $M(f)\vec{v} = \lambda\vec{v}$ we obtain directly

$$\vec{v}_m = M(f_{m,n})\vec{v}_n$$

for any integers $n \geqslant m \geqslant 0$, so that the family $\overleftarrow{v} := (\vec{v}_n)_{n\in\mathbb{N}\cup\{0\}}$ is a vector tower on $\overleftarrow{\Gamma}_f$.

Using Proposition 8.7 we now consider the image vector tower $\overleftarrow{g}(\overleftarrow{v}) =: (\vec{v}\,'_n)_{n\in\mathbb{N}\cup\{0\}}$ and observe that, by the above definition of $\overleftarrow{g}$, we have $\vec{v}\,'_n = M(g_n)\vec{v}_n$ for all integers $n \geqslant 0$. Since $g_n : \Gamma_n \to \Gamma_n$ is, via the identification $\Gamma_n = \Gamma$, equal to the map f, we obtain directly $\vec{v}\,'_n = \lambda\vec{v}_n$ for all $n \geqslant 0$, and hence via Proposition 8.4 the equality $\mu^{\overleftarrow{g}(\overleftarrow{v})} = \lambda\mu^{\overleftarrow{v}}$. Thus we obtain from Proposition 8.8 directly the following:

Proposition 9.4. *Let* $f : \Gamma \to \Gamma$ *be an expanding train track map that represents* $\varphi \in$ *Out*(F_N)*, with transition matrix* $M(f)$*.*

For any non-negative eigenvector $\vec{v}$ *of* $M(f)$ *with eigenvalue* $\lambda > 1$ *the current* $\mu^{\vec{v}} := \mu^{\overleftarrow{v}} \in$ *Curr*(F_N)*, defined by the vector tower* $\overleftarrow{v} = (\frac{1}{\lambda^n}\vec{v})_{n\in\mathbb{N}\cup\{0\}}$*, has support in* $L^{F_N}_\infty(f)$ *and satisfies:*

$$\varphi(\mu^{\vec{v}}) = \lambda\mu^{\vec{v}}$$

□

For the converse direction we consider a current $\mu \in \text{Curr}(F_N)$ which has support in $L^{F_N}_\infty(f) = L^{F_N}_{legal}(\overleftarrow{\Gamma}_f)$. Using Proposition 7.4 and the marking isomorphism $\pi_1\Gamma \cong F_N$, by

part (c) of Proposition 9.1 (2) the current μ defines a weight tower $\overleftarrow{\omega}^{\mu} = (\omega_n^{\mu})_{n \in \mathbb{N} \cup \{0\}}$ on $\overleftarrow{\Gamma}_f$, with $\mu^{\overleftarrow{\omega}^{\mu}} = \mu$. From the definition of the weight function ω_n^{μ} and the uniqueness statement in Proposition 7.4 one obtains directly $\overleftarrow{\omega}^{\lambda\mu} = (\lambda\omega_n^{\mu})_{n \in \mathbb{N} \cup \{0\}}$ for any scalar $\lambda > 0$. In particular, the associated vector towers $\overleftarrow{v}^{\mu} := (\vec{v}^{\omega_n^{\mu}})_{n \in \mathbb{N} \cup \{0\}}$ and $\overleftarrow{v}^{\lambda\mu} := (\vec{v}^{\lambda\omega_n^{\mu}})_{n \in \mathbb{N} \cup \{0\}}$ satisfy $\vec{v}^{\lambda\omega_n^{\mu}} = \lambda\vec{v}^{\omega_n^{\mu}}$ for any integer $n \geqslant 0$.

Proposition 9.5. *Let $f : \Gamma \to \Gamma$ be an expanding train track map that represents $\varphi \in Out(F_N)$, with transition matrix $M(f)$.*

For any current $\mu \in Curr(F_N)$, which has support in $L_{\infty}^{F_N}(f)$ and satisfies $\varphi(\mu) = \lambda\mu$ for some scalar $\lambda > 1$, there exists a non-negative eigenvector $\vec{v}$ of $M(f)$ with eigenvalue λ that satisfies

$$\mu = \mu^{\vec{v}}\,,$$

where $\mu^{\vec{v}}$ denotes the current $\mu^{\vec{v}} := \mu^{\overleftarrow{v}}$ defined by the vector tower $\overleftarrow{v} = (\frac{1}{\lambda^n}\vec{v})_{n \in \mathbb{N} \cup \{0\}}$.

Proof. From the hypothesis $\varphi(\mu) = \lambda\mu$ we deduce that the vector tower $\overleftarrow{v}^{\lambda\mu}$ considered in the paragraph before the proposition must agree with the image vector tower $\overleftarrow{g}(\overleftarrow{v}^{\mu})$ of $\overleftarrow{v}^{\mu}$ under the graph tower self-morphism $\overleftarrow{g} : \overleftarrow{\Gamma}_f \to \overleftarrow{\Gamma}_f$, induced by the train track map f as spelled out above. Thus, using for any level $n \geqslant 0$ the fact that the level map g_n is precisely given (via the identifications $\Gamma_n = \Gamma$) by the train track map f, we deduce for the above description of the vector towers $\overleftarrow{v}^{\mu} := (\vec{v}^{\omega_n^{\mu}})_{n \in \mathbb{N} \cup \{0\}}$ and $\overleftarrow{v}^{\lambda\mu} := (\vec{v}^{\lambda\omega_n^{\mu}})_{n \in \mathbb{N} \cup \{0\}}$, with $\vec{v}^{\lambda\omega_n^{\mu}} = \lambda\vec{v}^{\omega_n^{\mu}}$, that

$$\vec{v}^{\lambda\omega_n^{\mu}} = \lambda\vec{v}^{\omega_n^{\mu}} = M(f)\vec{v}^{\omega_n^{\mu}}$$

for any $n \geqslant 0$, or in other words: each $\vec{v}^{\omega_n^{\mu}}$ is an eigenvector $\vec{v}_n$ of $M(f)$ with eigenvector λ. We now use the fact that the graph maps $f_{n,n+1}$ of the graph tower $\overleftarrow{\Gamma}_f$ are (via the identification $\Gamma_n = \Gamma = \Gamma_{n+1}$ identical to the train track map f. Hence the compatibility equalities in Definition 8.2 imply that $\vec{v}_{n+1} = \frac{1}{\lambda}\vec{v}_n$ for all $n \geqslant 0$. This shows that the vector tower $\overleftarrow{v}^{\mu}$ agrees indeed with the vector tower $\overleftarrow{v} = (\frac{1}{\lambda^n}\vec{v})_{n \in \mathbb{N} \cup \{0\}}$ defined above for the eigenvector $\vec{v} := \vec{v}_0$. As direct consequence we obtain $\mu^{\overleftarrow{v}} = \mu^{\overleftarrow{v}^{\mu}} = \mu$, which is the claim of the proposition. □

10. Invariant measures on the subshift defined by a train track map

In this section we want to consider train track maps $f : \Gamma \to \Gamma$ that are not necessarily homotopy equivalences. In this case, if Γ is provided with a marking, f induces an endomorphism of F_N which is possibly non-injective. The translation of the dynamics of f into the $\partial^2 F_N$-setting of algebraic laminations and currents as given in [7], [8] is problematic, as a non-injective endomorphism does not even yield a well defined self-map of ∂F_N.

As a consequence, contrary to what has been done in the previous chapters, in this section we will not transfer the combinatorial data given by the train track map f via a marking to a free group F_N or its double boundary $\partial^2 F_N$. The reason is that, as explained in detail in subsection 2.7, in spite of the fact that f does in general not induce a well defined projectively f-invariant current on F_N, one still obtains a well defined invariant measure on the symbolic dynamics "subshift" space $\Sigma(\Gamma)$ defined combinatorially for the graph Γ.

We first define a property for train track maps $f : \Gamma \to \Gamma$ which is automatically satisfied for homotopy equivalences. Recall that a reduced edge path γ in Γ is f-*legal* if for any integer

$t \geqslant 1$ the path $f^t(\gamma)$ is reduced, and γ is *infinitely f-legal* if for any $t \geqslant 1$ the path γ is a subpath of $f^t(\gamma_t)$ for some f-legal path γ_t.

Definition 10.1. An expanding train track map $f : \Gamma \to \Gamma$ is called *non-repeating*, if for any $n \in \mathbb{N}$ there exists a *repetition bound* $\rho_f(n) \in \mathbb{N}$ which has the following property:

Any two infinitely f-legal paths γ and γ' in Γ of length $|\gamma| = |\gamma'| = 2\rho_f(n) + 1$ which satisfy $f^n(\gamma) = f^n(\gamma')$ have coinciding middle edge.

Remark 10.2. The property "non-repeating" can alternatively be understood as follows: Recall from subsection 2.7 that the train track map f defines canonically a symbolic lamination (= a subshift) $L^{\Sigma}_{\infty}(f)$ which is equipped with a shift map $S : L^{\Sigma}_{\infty}(f) \to L^{\Sigma}_{\infty}(f)$ and has the set of infinitely f-legal paths $\mathcal{L}_{\infty}(f)$ as associated laminary language. The train track map f induces a map on $L^{\Sigma}_{\infty}(f)$, and also on the quotient set $L^{\Sigma}_{\infty}(f)/\langle S\rangle$ of S-orbits. It is not difficult to show that the induced map $f^{\Sigma/S}$ on this quotient space is always surjective.

An easy diagonal argument now shows that the property "non-repeating" amounts precisely to stating that the map $f^{\Sigma/S}$ is bijective.

The authors are at present not aware of any example of an expanding train track map which does not have a biinfinite periodic infinitely f-legal path, and is repeating.

We now consider again the graph tower $\overleftarrow{\Gamma}_f$ defined by a train track map $f : \Gamma \to \Gamma$ as described in section 9. Recall from Remark 9.2 (which applies also to train track maps f that are not homotopy equivalences) that in this situation the laminary language $\mathcal{L}_{\infty}(f)$ of infinitely f-legal paths defined by the train track map coincides with the set $\mathcal{P}_{legal}(\overleftarrow{\Gamma}_f)$ of infinitely legal paths with respect to the graph tower $\overleftarrow{\Gamma}_f$, and one has:

$$f(\mathcal{L}_{\infty}(f)) \subseteq \mathcal{L}_{\infty}(f)$$

The set $\mathcal{L}_{\infty}(f)$ is clearly contained in the set $\mathcal{L}(f)$ of paths in Γ that have reduced f-images (compare subsections 2.6 and 2.7), so that the corresponding subshift $L^{\Sigma}_{\infty}(f)$ is contained in $L^{\Sigma}(f)$. In particular, using Definition 2.2 we see that every invariant measure μ_{Σ} on $L^{\Sigma}_{\infty}(f)$ possesses a well defined f-image $f_*\mu_{\Sigma}$, which has support in $L^{\Sigma}_{\infty}(f) \subseteq L^{\Sigma}(f)$ (by the above inclusion $f(\mathcal{L}_{\infty}(f)) \subseteq \mathcal{L}_{\infty}(f)$).

A graph tower $\overleftarrow{\Gamma}$ is called *weakly non-repeating* if in Definition 7.1 the paths γ and γ' are not only assumed to be legal, but actually infinitely legal. Since in Proposition 7.4 one assumes that the support of the invariant measure μ_{Σ} is contained in the infinitely legal lamination $L^{\Sigma}_{legal}(\overleftarrow{\Gamma})$, the given proof stays valid for this slightly weakened assumption.

It follows directly from the definitions that for any non-repeating train track map $f : \Gamma \to \Gamma$ the issuing vector tower $\overleftarrow{\Gamma}_f$ is weakly non-repeating.

As a consequence, we see that Proposition 9.1 can be used as in the previous section, since the hypothesis in statement (c) that f be a homotopy equivalence is now replaced by "f non-repeating". Thus proceeding exactly as in the previous section yields the following, which also proves Theorem 1.3 from the Introduction:

Theorem 10.3. *Let $f : \Gamma \to \Gamma$ be an expanding non-repeating train track map (not necessarily a homotopy equivalence), and let $\lambda > 1$ be an eigenvalue of $M(f)$.*

There is a canonical bijection between the set $\mathcal{M}_{\lambda}(f)$ of finite invariant measures μ_{Σ} on $L^{\Sigma}_{\infty}(f)$ which satisfy $f_*\mu_{\Sigma} = \lambda\mu_{\Sigma}$, and the set $V_{\lambda}(f)$ of non-negative eigenvectors $\vec{v}$ of $M(f)$

with eigenvector λ. This bijection is given by

$$\vec{v} \mapsto \mu_\Sigma := \mu_\Sigma^{\vec{v}}\,,$$

where $\mu_\Sigma^{\vec{v}}$ denotes the invariant measure on $L_\infty^\Sigma(f)$ with associated Kolmogorov function $\mu_\Gamma^{\vec{v}}$ that is defined by the vector tower $\overleftarrow{v} = (\frac{1}{\lambda^n}\vec{v})_{n\in\mathbb{N}\cup\{0\}}$ on the graph tower $\overleftarrow{\Gamma}_f$ for f. □

11. The results of S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak for substitutions

In this section we compare our approach with results obtained by S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak [4] for *substitutions* in symbolic dynamics. We will use freely the standard terminology as it has been reviewed in subsection 2.8. In order to be able to state the result, we will first recall quickly some folklore facts from Perron-Frobenius theory for non-negative integer matrices.

We recall that a non-negative integer square matrix M is called *reducible* if it can be written (through conjugation with a permutation matrix) as upper triangular block matrix with 2 or more diagonal blocks. Otherwise it is called *irreducible*. The matrix M is *primitive* if there exists an integer $k \geqslant 1$ such that every coefficient of M^k is positive. Every primitive integer matrix M has a positive *PF-eigenvector* $\vec{v}$ with associated *PF-eigenvalue* $\lambda \geqslant 1$ which is equal to the spectral radius of M (and $\vec{v}$ is, up to rescaling, the only eigenvector of M with this last property).

Up to conjugation with a permutation matrix every non-negative matrix can be written as upper triangular block matrix $M = (A_{i,j})_{i,j}$, such that every diagonal block $A_{i,i}$ is irreducible. Through replacing M by a positive power, we can furthermore assume that each $A_{i,i}$ is either primitive, or else a 1×1 zero matrix. In this case, through possibly passing to a further positive power, one can achieve that every off-diagonal block $A_{i,j}$ is either *zero* (i.e. all coefficients are equal to 0), or *positive* (i.e. all coefficients are strictly bigger than 0).

The matrix M defines a canonical partial order on the diagonal blocks (assumed to be primitive or zero) through defining $A_{i,i} \succeq A_{j,j}$ if for a suitable power of M the off-diagonal block $A_{j,i}$ is positive. We say that $A_{i,i}$ is *distinguished* if $\lambda_i \neq 0$ and $\lambda_i > \lambda_j$ for all $A_{j,j}$ with $A_{i,i} \succeq A_{j,j}$, where λ_k denotes the spectral radius of $A_{k,k}$. It is part of standard Perron-Frobenius theory that to every distinguished diagonal block $A_{i,i}$ there is precisely one *distinguished* eigenvector $\vec{v}_i$ of M with eigenvalue λ_i. By this we mean that $\vec{v}_i$ is non-negative, agrees on $A_{i,i}$ (up to rescaling) with the corresponding PF-eigenvector, and is normalized so that the sum of its coordinates is equal to 1.

The results of [4] that we are considering here are stated in Theorem 11.1 below. It results from earlier, more general work of the authors, and its proof is heavily based on the use of Bratteli Diagrams, Vershik maps, and other non-elementary techniques from symbolic dynamics (see [3]).

Theorem 11.1 (Bezuglyi, Kwiatkowski, Medynets, Solomyak, Corollary 5.6 of [4])**.** *Let $\mathcal{A}$ be a finite alphabet, and let σ be a substitution over $\mathcal{A}$. We assume that $\lim_{+\infty} |\sigma^n(a)| = +\infty$ for every letter $a \in \mathcal{A}$, and that the subshift X_σ defined by σ does not contain a periodic word.*

Up to replacing σ by a positive power σ^k (so that every diagonal block is primitive), the set of ergodic probability measures for the subshift X_σ is in 1 - 1 correspondence with the set of distinguished eigenvectors for M_σ.

Using the translation between classical symbolic dynamics and symbolic dynamics on graphs as explained in Remark 2.5, we see that indeed there is a strong relationship between the above theorem and our Theorem 10.3. There are, however, several subtle differences, which we would like to point out now:

(1) The main difference is that in Theorem 10.3 eigenvectors of $M(f)$ are in correspondence with shift-invariant measures μ_Σ that are projectively invariant under the graph map f, while in the theorem above eigenvectors are related to ergodic measures without direct relationship to the substitution.

(2) A second difference is that the information of which invariant measure precisely corresponds to a given eigenvector $\vec{v}$ of M_σ is less directly available in [4], in the sense that it has to be first transduced via a Bratteli diagram, then investigated, and finally transduced back. Indeed, although it is of course expected that this invariant measure coincides indeed with the measure $\mu_\Sigma^{\vec{v}}$ (after proper translation through Remark 2.5), we have so far not been able to extract this information in full formality.

(3) A third difference is that in Theorem 10.3 we need as extra-assumption on the train track map that it is non-repeating, while [4] only requires that σ is expanding and that the associated subshift X_σ doesn't have periodic words. We do not know at present whether every expanding train track map f without periodic words in the associated subshift $L^\Sigma_\infty(f)$ is non-repeating, but there are indications that this is indeed true.

(4) At first sight the approach of [3, 4] seems to be weaker in that it doesn't apply to arbitrary train track maps and thus only captures what is known as "positive" automorphisms or endomorphisms of a free group. However, there is a known technology how to transfer train track maps into the setting of substitutions (see [1] for instance), and it is not impossible that via this translation one can recover the full realm of Theorem 10.3 by the above Theorem 11.1.

(5) The fact that in Theorem 11.1 one considered only distinguished eigenvectors is due to the fact that there one considers only ergodic measures. Indeed, every non-negative eigenvector of a non-negative matrix M is a unique linear combination of the distinguished eigenvectors (of same eigenvalue) for the corresponding power M^k.

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